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Capital-Labor Substitution, Structural Change and Growth

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ABSTRACT

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There is a growing interest in multi-sector models that combine aggregate balanced growth, consistent with the well-known Kaldor facts, with systematic changes in the sectoral allocation of resources, consistent with the Kuznets facts. Although variations in the income elasticity of demand across goods played an important role in initial approaches, recent models stress the role of supply-side factors in this process of structural change, in particular sector-specific technical change and sectoral differences in factor proportions. We explore a general framework that features an additional supply-side mechanism and also encompasses, as special cases, these two known mechanisms. Our model shows that sectoral differences in the degree of capital-labor substitutability – a new mechanism – are a driving force for structural change. When the flexibility to combine capital and labor differs across sectors, a factor rebalancing effect is operative. It tends to make production in the more flexible sector more intensive in the input that becomes more abundant. As a result, growth rates of sectoral capital-labor ratios can differ and, if this effect dominates, shares of each factor used in a given sector can move in different directions. We identify conditions under which this occurs and analyze the dynamics of such a case. We also provide some suggestive evidence consistent with this new mechanism.

JEL Classification: O40, 041, O30

Keywords: capital-labor substitution, balanced growth, structural change

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1 Introduction

The theoretical literature on economic growth has traditionally been interested in models that exhibit a balanced growth path, a trajectory where the growth rate of output, the capital-output ratio, the return to capital, and factor income shares are constant. It has become standard to impose restrictions on preferences and technology to be consistent with these "Kaldor facts" (Kaldor, 1963). Nonetheless, underlying this balanced process at the aggregate level, there are systematic changes in the composition of output at a more disaggregated level – a secular process of structural change. The seminal work of Clark (1940) and Kuznets (1966) already documented a facet of this structural transformation, particularly the continuous decrease in the share of agriculture in output and employment that accompanies long-run increases in income per capita. More recently, several authors (see for instance Kongsamut et al. 2001; Buera and Kaboski, 2012) have drawn attention to the increasing importance of the service sector. Kongsamut et al. (2001) have dubbed this second set of empirical regularities associated with the process of structural change the "Kuznets facts".

In recent years, several multi-sector growth models that address both the Kaldor and the Kuznets facts have been proposed. Inspired by the early contributions of Baumol (1967) and Matsuyama (1992), this literature has identified several channels that can drive structural change and are still consistent with a balanced growth path. One can classify these channels into two categories: preference-driven and technology-driven structural change.[1] In the first category (see for instance Kongsamut et al. (2001), Foellmi and Zweimüller (2008) or Boppart (2014)), structural change is driven by differences in the income elasticity of demand across goods.[2] As capital accumulates and income rises, these differences shift demand, and therefore resources and production, from goods with low demand elasticity, such as food or necessities, to high demand elasticity goods, such as services or luxuries. In the second category, where technological differences across sectors play the dominant role, two alternative mechanisms have been proposed. The first mechanism, recently formalized by Ngai and Pissarides (2007), hereafter NP, works through differences in the sectoral rates of TFP growth. The second mechanism, explored by Acemoglu and Guerrieri (2008), hereafter

[1] Most papers in this literature use a closed economy framework. In this context, the interaction between preferences and technology determines sectoral allocations. In an open economy, given world prices, sectoral allocations are only determined by the supply side of the model. See Matsuyama (1992) and Ventura (1997) for models of structural change in an open economy and Alvarez-Cuadrado and Poschke (2011) for the relevance of the closed-economy assumption in the context of structural change out of agriculture.

[2] There is a large body of work that assumes non-homotheticity as a source for structural change. See also Echeverria (1997), Laitner (2000), Caselli and Coleman (2001), Gollin, Parente and Rogerson (2007), and Restuccia and Duarte (2010). Boppart (2014) considers non-homothetic preferences and differential TFP growth jointly.

AG, places its emphasis on sectoral differences in factor proportions, i.e. differences in the elasticity of output with respect to capital across sectors. As sectoral levels of TFP diverge or capital accumulates, these two channels generate a process of structural change.[3]

The main objective of this paper is to explore an additional source for technology-driven structural change consistent with quasi-balanced growth at the aggregate level: sectoral differences in the elasticity of substitution between capital and labor. Intuitively, if the degree of flexibility with which capital and labor can be combined to produce output varies across sectors, changes in the wage to rental rate ratio that accompany aggregate growth lead to systematic reallocations of factors of production and to changes in the sectoral composition of output. In this paper, we formalize this simple intuition. In our model, as the aggregate capital-labor ratio and the wage to rental rate ratio increase, the sector with higher elasticity of factor substitution – the more flexible sector – is in a better position to substitute away from the progressively relatively more expensive input, labor, and into the progressively cheaper one, capital, compared to the less flexible sector. As a result, differences in the sectoral elasticity of substitution between capital and labor induce a process of structural change.

Our exercise is motivated by three observations. Firstly, there is direct econometric evidence that the elasticity of substitution differs across sectors. Most recently, Herrendorf, Herrington and Valentinyi (2014) have estimated sectoral CES production functions using postwar U.S. data. Their results indicate not only substantial deviations from the Cobb-Douglas benchmark, but also pronounced differences across sectors, with an elasticity of substitution between capital and labor below one for both services and manufacturing, and an elasticity well above one for agriculture. Secondly, there is evidence that capital-labor ratios grow at different rates in different sectors. Most notably, the yearly average growth rate of the capital-labor ratio in U.S. agriculture has been 3.2% since the end of World War II, twice as large as that in non-agriculture.[4] As we will see, under free factor mobility, this cannot arise if sectoral production functions are Cobb-Douglas, and thus also points towards differences in substitution elasticities across sectors. Thirdly, factor income shares have evolved differently across sectors. For instance, Zuleta and Young (2013) construct sectoral labor income shares using data from the U.S. 35 sector KLEM database from 1960

[3]Baumol (1967) suggests several mechanisms behind structural change, specifically "innovations, capital accumulation, and economies of large scale" (p. 415). Recently, Buera and Kaboski (2012) developed a model where structural change results from differences in the scale of productive units across sectors. This is yet another source of technology-driven structural change.

[4]See NIPA Fixed Asset Tables 3.1ESI and 3.2ESI and Income and Employment by Industry Tables 6.5A-D, 6.7A-D and 6.8A-D, years 1947 to 2012.

to 2005 (Jorgenson, 2007). Over this period, the labor income share in agriculture fell by 15 percentage points, roughly three times its change in the rest of the economy.[5] While factor income shares have a whole set of determinants, it is clear that this observation is inconsistent with Cobb-Douglas production functions at the sectoral level, and also points towards sectoral differences in factor substitutability.

These changes in sectoral capital-labor ratios and factor income shares coincided with substantial structural change. For instance, the contribution of agricultural value added to U.S. GDP declined by 90% from the end of World War II to 2000, from almost 10% to just 1% of GDP. Over the same period, the fraction of persons engaged in agriculture declined from 14% to barely 2%. The fraction of fixed assets used in agriculture only declined from 4% to 1.6%, illustrating again the much faster growth in the capital-labor ratio in agriculture. The crucial role played by the elasticity of substitution in the evolution of the capital-labor ratios and the factor income shares suggests that this elasticity may also play an important role in the process of structural change.[6]

We thus analyze theoretically how differences in factor substitutability affect structural change. The framework we use for our analysis is a two-sector version of the Solow model. Final output is produced using a CES aggregator that combines two intermediate inputs. These are in turn sectoral outputs produced under two different CES production functions, using capital and labor. By varying parameter restrictions, this simple framework allows us not only to analyze the new mechanism that we are proposing, but also to capture the essence of the two supply-side mechanisms stressed in the previous literature.

To begin, we identify conditions on parameters that determine how factor allocations react to capital accumulation and technical change. These conditions arise from the balance of three effects: a *relative price effect*, a *relative marginal product effect*, and *a factor rebalancing effect*. While the former two are already present in the previous literature on technology-driven structural change, the latter is new and arises only in the presence of dif-

[5] Aside from the high frequency variations in factor income shares, documented for instance by Blanchard (1997), Caballero and Hammour (1998) and Bentolila and Saint-Paul (2003), there has been a recent reappraisal of the long-run constancy of the aggregate labor income share. Two recent contributions, Elsby, Hobijn, and Sahin (2013) and Karabarbounis and Neiman (forthcoming), have documented a secular decline in the labor income share in the United States and at a global level. In a companion paper, Alvarez-Cuadrado, Long, and Poschke (2014), we explore the implications of sectoral differences in the elasticity of substitution between capital and labor for this decline in the aggregate labor income share.

[6] Recall that this elasticity was first introduced by Hicks in his seminal work, *The Theory of Wages* (1932), to explore the distribution of income between factors in a growing economy. First, Pitchford (1960) and, recently, Klump and de La Granville (2000) and Jensen (2003) analyze the relationship between the CES production function and the possibility of permanent growth in a neoclassical model of capital accumulation. The latter explores multi-sector models, but differs in focus from our work.

ferences in the elasticity of substitution across sectors. It reflects that, as a factor becomes more abundant and therefore relatively cheaper, e.g. due to capital accumulation, the more flexible sector increases its use of that factor more. We show that if this effect is strong enough, it is possible for sectoral capital and labor allocations to move in opposite directions with capital accumulation or technical change. More generally, its presence leads to differences in the evolution of capital-labor ratios. The evolution of value added shares of the two sectors depends on whether an increase in the capital-labor ratio makes the economy's endowments of capital and labor more or less "balanced."

We then analyze the dynamics of three special cases, which focus on the driving channels in the AG and NP models, respectively, and on the new factor rebalancing effect. A first case, along the lines of the AG model, arises when sectoral outputs are produced using Cobb-Douglas technologies that differ in capital intensity and the elasticity of substitution in the production of final output differs from one. For instance, it could be less than one, so that the two intermediate inputs are gross complements. In this case, as capital accumulates, the fractions of capital and labor allocated to the labor-intensive sector increase. Intuitively, if the fraction of resources allocated to each sector were kept constant, output in the capital-intensive sector would grow faster. This imbalance would cause a drop in the relative price of the output of that sector, prompting a shift of resources (labor and capital) into production of the labor-intensive good.

A second case, a simplified version of the NP model, arises when the elasticity of substitution in the production of final output differs from one, for instance is less than one, and sectoral outputs are produced using identical Cobb-Douglas technologies with sector-specific rates of TFP growth.[7] For the sake of illustration, let the growth rate be larger in sector 2. Then, the fractions of capital and labor allocated to sector 1 will increase over time. Intuitively, if the fractions of capital and labor allocated to each sector remained constant, output would grow faster in sector 2. The two intermediate goods being complements in the production of the final good, this would again prompt a more than proportionate decline in the relative price of good 2 – a version of Baumol's cost disease. As a result, capital and labor must move into the technologically laggard sector 1.[8]

[7] Ngai and Pissarides (2007) includes at least two important contributions. First, they explore the implications of differences in sectoral TFP growth rates for structural change. Second, they derive conditions for the co-existence of structural change and balanced growth. In contrast to our simplified version of the NP model where capital is accumulated out of final output, in their original model these authors distinguish between consumption and capital-producing sectors. This distinction, which our simplified version abstracts from, turns out to be crucial for their second result.

[8] In both the simplified NP model and the AG model, a similar but opposite argument applies when the elasticity of substitution of the final sector exceeds unity, i.e. when sectoral outputs are gross substitutes in

A third case we consider allows the analysis of the novel mechanism for structural change that relies on differences in the substitution elasticity. This case arises when the factor rebalancing effect is strong enough. To make the analysis of dynamics tractable, we impose more assumptions, namely that final output is produced under a Cobb-Douglas technology and the sectoral technologies are identical except for the sectoral elasticity of substitution between capital and labor. For expositional purposes, let the elasticity of substitution of sector 2 exceed that of sector 1, making sector 2 the flexible sector. In this case, as the aggregate capital-labor ratio increases, the fractions of capital and labor allocated to the flexible sector move in opposite directions: the flexible sector absorbs more capital and releases labor. Intuitively, as capital accumulates and the wage to rental rate ratio increases, the flexible sector will tend to substitute from the now more expensive input, labor, towards the relatively cheaper one, capital, at a higher rate than the less flexible sector 1 is able to do. Sectoral capital-labor ratios and factor income shares thus can evolve differently in the two sectors. In the context of this simple framework, we show that the economy converges to a balanced growth path consistent with the Kaldor facts, with structural change taking place along the transition only. These results, derived under a constant saving rate, readily extend to an optimal saving environment.

It is worth stressing that the differences in the degree of factor substitution are a distinct driver of structural change compared to the differences in factor proportions stressed in the AG model. This is true both in terms of implications, as just discussed, and conceptually. Conceptually, for given factor prices, factor proportions are determined by the interaction between the elasticity of substitution and the distributive parameter (the α in the Cobb-Douglas technology). AG focus on sectoral differences in this latter parameter, while our model stresses the effects of differences in the former.

Finally, the mechanism proposed here also differs from those in AG and NP in that structural change is driven by changes in the relative price of factors, not the relative price of sectoral outputs. As made clear in the arguments above, in both the AG and NP models, the factor allocation changes with growth in response to changes in the relative price of the two intermediate goods. Instead, with differences in the elasticity of substitution, the factor allocation changes in response to changes in the ratio of the wage rate to the rental rate of capital.

The mechanism illustrated in this paper is related to that in Ventura (1997) and to the literature on capital-skill complementarity initiated by Krusell, Ohanian, Rios-Rull and

the production of the final output.

Violante (2000). Ventura (1997) presents a multi-country growth model where final output is produced combining two intermediate goods, one of which is produced using only capital, while the other uses only labor. There is free trade in both intermediate goods, although international factor movements are not permitted. In this context, as a country accumulates capital, resources are moved from labor-intensive to capital-intensive uses – a process of structural change – while international trade converts this excess production of capital-intensive goods into labor-intensive ones. Krusell et al. (2000) present a neoclassical growth model where the elasticity of substitution between capital equipment and unskilled labor is higher than that between capital equipment and skilled labor. As a result, as capital accumulates, the wage for skilled workers increases relative to that of unskilled workers, in line with the increase in the skill premium observed over the last 20 years of the past century.

The paper is organized as follows. Section 2 sets out the basic model. Section 3 analyzes optimal static allocations and explores the three cases we have just discussed. Section 4 analyzes growth paths for these three cases, and Section 5 presents the solution of the model with unequal sectoral capital-labor substitution under optimal saving. Section 6 provides some suggestive evidence and discusses some additional issues. The conclusions are summarized in section 7, while the Appendices include proofs and some technical details.

2 A general model of structural change

We model a closed economy where a single final good is produced under perfect competition by combining the output of two intermediate-good sectors, Y_s, where $s = 1, 2$, according to a CES technology with elasticity of substitution $\varepsilon \in [0, \infty)$:

$$Y(t) = F(Y_1(t), Y_2(t)) = \left[\gamma Y_1(t)^{\frac{\varepsilon-1}{\varepsilon}} + (1-\gamma) Y_2(t)^{\frac{\varepsilon-1}{\varepsilon}}\right]^{\frac{\varepsilon}{\varepsilon-1}} \quad (1)$$

where $\gamma \in (0, 1)$ is the distributive share.[9] Both intermediate-good sectors use two inputs, labor, L, and capital, K. The labor force grows at a rate n and capital depreciates at a rate δ. The aggregate resource constraint requires the sum of consumption, C, and gross investment, I, to be equal to output of the final good.

$$\dot{K}(t) + \delta K(t) + C(t) \equiv I(t) + C(t) = Y(t) \quad (2)$$

where the dot denotes the change in a variable. Under the assumption that a fixed fraction of output, s, is saved and invested every period, equation (2) yields the following law of

[9] When $\varepsilon = 1$, this equation becomes $Y = Y_1^{\gamma} Y_2^{1-\gamma}$.

motion for the capital stock,

$$\dot{K}(t) = sY(t) - \delta K(t). \tag{3}$$

The two intermediate goods are produced competitively according to

$$Y_s(t) = \left[(1-\alpha_s)(M_s(t)L_s(t))^{\frac{\sigma_s-1}{\sigma_s}} + \alpha_s K_s(t)^{\frac{\sigma_s-1}{\sigma_s}}\right]^{\frac{\sigma_s}{\sigma_s-1}} \tag{4}$$

where $\alpha_s \in (0,1), \sigma_s \in [0,\infty), M_s, L_s,$ and K_s are respectively the distributive share, the elasticity of substitution, and the levels of technology, employment, and capital for sector s.[10] Both inputs are fully utilized:

$$\begin{aligned} L_1(t) + L_2(t) &= L(t), \\ K_1(t) + K_2(t) &= K(t). \end{aligned}$$

Technological progress in each sector is exogenous and exhibits a constant growth rate:

$$\frac{\dot{M}_s(t)}{M_s(t)} = m_s \geqslant 0, \quad s = 1, 2. \tag{5}$$

This general setup includes, as special cases, several of the mechanisms described in the theoretical literature on structural change. First, AG present a model of structural change (and non-balanced growth) driven by differences in factor proportions and capital deepening. The distinctive features of their model are recovered by setting $\varepsilon \neq 1$, $\sigma_1 = \sigma_2 = 1$, $m_2 = m_1$ and $\alpha_1 \neq \alpha_2$. Second, NP present a model where structural change is driven by differences in the growth rates of sectoral TFP. A simplified version of their model is recovered setting $\varepsilon \neq 1$, $\sigma_1 = \sigma_2 = 1$, $\alpha_1 = \alpha_2$ and $m_2 \neq m_1$. Third, we show below the presence of a novel mechanism based on differences in the degree of capital-labor substitutability across sectors.

We break the solution of our problem into two steps. First, given the vector of state variables at any point in time, (K, L, M_1, M_2), the allocation of factors across sectors is chosen to maximize final output, (1). This is the static problem, solved in the next section. Second, given factor allocations at each date, the time path of the capital stock follows the law of motion (3). The dynamic problem consists of examining the stability of this process. This is analyzed in Section 4. Finally, Section 5 explores an optimal growth version of the model.

[10] Again, the possibility that $\sigma_s = 1$ is admitted, for one or for both sectors. Also note that the capital income share in sector s is not simply equal to α_s unless $\sigma_s = 1$.

3 The static problem

Let us denote the rental rate, the wage rate, the prices of the intermediate goods and the price of the final good by $R \equiv r + \delta$, w, p_1, p_2 and P respectively.[11] It will prove useful to define capital per worker, $k = K/L$, and the shares of capital and labor allocated to sector 1 as

$$\kappa(t) \equiv \frac{K_1(t)}{K(t)} \quad \text{and} \quad \lambda(t) \equiv \frac{L_1(t)}{L(t)}. \tag{6}$$

Throughout, we will also assume without loss of generality that $\sigma_2 > \sigma_1$, i.e. sector 2 is the more flexible sector.

The optimal allocation involves two trade-offs: the optimal allocation of resources across sectors, and the optimal balance of resources, or the optimal capital-labor ratio, in each sector. The former is summarized by the *labor mobility condition (LM)* and the latter by the *contract curve (CC),* which we derive and characterize next.

3.1 The contract curve (CC)

At any point in time and for any prices, free mobility of capital and labor across sectors implies the equalization of the marginal value products of these factors across sectors:

$$p_1 \alpha_1 \left(\frac{Y_1}{K_1}\right)^{\frac{1}{\sigma_1}} = p_2 \alpha_2 \left(\frac{Y_2}{K_2}\right)^{\frac{1}{\sigma_2}} = R \tag{7}$$

$$p_1 (1-\alpha_1) \left(\frac{Y_1}{L_1}\right)^{\frac{1}{\sigma_1}} M_1^{\frac{\sigma_1 - 1}{\sigma_1}} = p_2 (1-\alpha_2) \left(\frac{Y_2}{L_2}\right)^{\frac{1}{\sigma_2}} M_2^{\frac{\sigma_2 - 1}{\sigma_2}} = w \tag{8}$$

Combining these equations yields the contract curve:

$$CC(\kappa, \lambda, k, M_1, M_2) \equiv \frac{(1-\alpha_1)}{(1-\alpha_2)} \frac{\alpha_2}{\alpha_1} \frac{M_1^{\frac{\sigma 1-1}{\sigma_1}}}{M_2^{\frac{\sigma_2 - 1}{\sigma_2}}} k^{\frac{1}{\sigma_1} - \frac{1}{\sigma_2}} \frac{\kappa^{\frac{1}{\sigma_1}}}{(1-\kappa)^{\frac{1}{\sigma_2}}} \frac{\lambda^{-\frac{1}{\sigma_1}}}{(1-\lambda)^{-\frac{1}{\sigma_2}}} = 1. \tag{CC}$$

This curve describes the set of points where the marginal rates of technical substitution are equalized across sectors.

To characterize it, it is useful to compute the elasticity of its left hand side, which we will also term (CC) in a slight abuse of notation, with respect to the key endogenous and

[11] We drop the time indicators when there is no risk of ambiguity.

forcing variables. These elasticities are given by

$$\epsilon_{CC,\lambda} = -\frac{1}{\sigma_1} - \frac{1}{\sigma_2}\frac{\lambda}{(1-\lambda)} < 0 \tag{9a}$$

$$\epsilon_{CC,\kappa} = \frac{1}{\sigma_1} + \frac{1}{\sigma_2}\frac{\kappa}{(1-\kappa)} > 0 \tag{9b}$$

$$\epsilon_{CC,k} = \frac{1}{\sigma_1} - \frac{1}{\sigma_2} > 0 \tag{9c}$$

$$\epsilon_{CC,M_1} = \frac{\sigma_1 - 1}{\sigma_1} \tag{9d}$$

$$\epsilon_{CC,M_2} = \frac{1-\sigma_2}{\sigma_2}, \tag{9e}$$

where the signs follow from the assumption that $\sigma_2 > \sigma_1$.

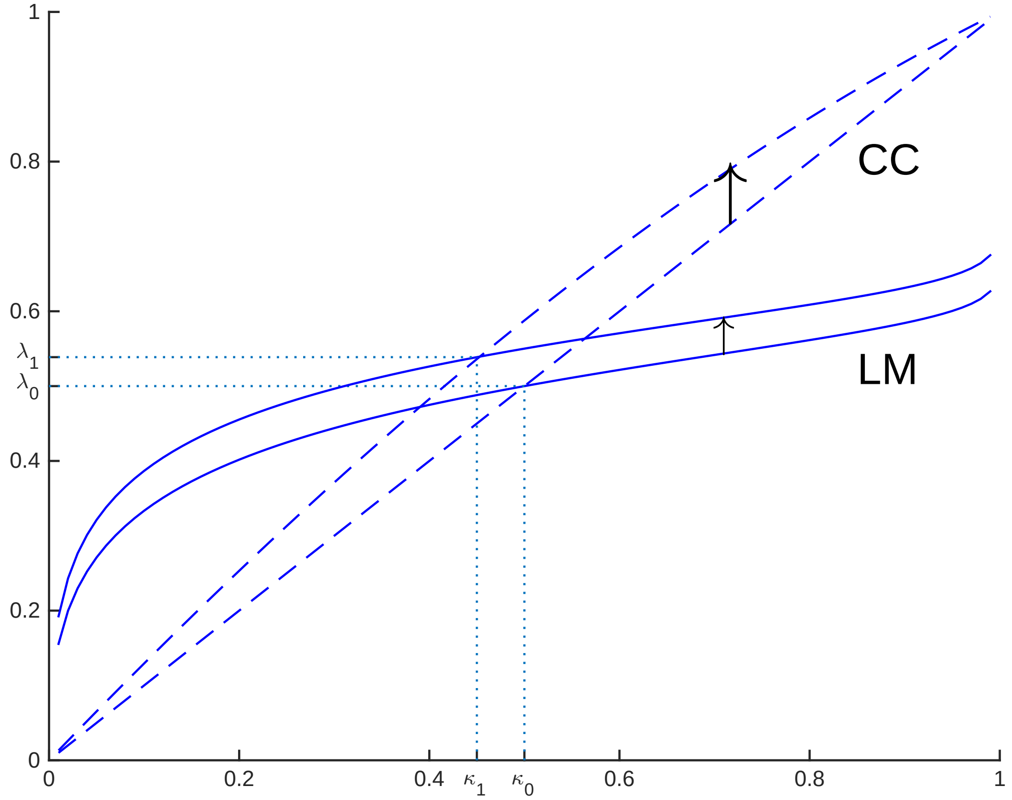
For the analysis of the optimal allocation of capital and labor across sectors and its shifts with development, it is useful to depict the contract curve and the labor mobility condition in κ, λ-space (see Figure 1). The elasticities of (CC) with respect to κ and λ imply that the curve is upward-sloping in this space. It is also clear from equation (CC) that it connects the origin with the point (1,1). (To see this, bring the fraction involving λ to the right hand side.)

The positive slope of the curve reflects the complementarity of capital and labor in production: Increasing the amount of capital allocated to sector 1 increases the marginal product of labor in sector 1 relative to sector 2, and therefore also calls for allocating more labor to sector 1. Note that, of course, the curve can also be transformed to show the optimal relationship between capital-labor ratios in the two sectors.

A few special cases are worth of notice. In the case where both sectors have Cobb-Douglas production functions (as in AG and NP), $\sigma_1 = \sigma_2 = 1$, and (CC) implies that $\kappa/(1-\kappa)$ is a linear function of $\lambda/(1-\lambda)$. (Note that this implies that k_1 and k_2 are proportional, but not that κ and λ are proportional.) In this case, (CC) reacts neither to changes in k nor to changes in M_1 or M_2. When in addition $\alpha_1 = \alpha_2$, $\kappa = \lambda$, and capital-labor ratios are equated across sectors. When σ is common (but not equal to one), $\kappa/(1-\kappa)$ and $\lambda/(1-\lambda)$ are also proportional, and unaffected by changes in k or by proportional changes in M_s in both sectors. They do however respond to changes in relative M_s. Capital intensity is higher in the sector with larger α_s or with larger (smaller) M_s, if $\sigma <$ (>)1.

It is clear from this discussion that in the Cobb-Douglas case, the contract curve is not very interesting, and all “action” in terms of structural change is driven by movements in the labor mobility curve. This is not true anymore when σ differs across sectors. In this case, movements in both (CC) and (LM) will jointly determine structural change. Shifts of

Figure 1: The contract curve (CC) and the labor mobility condition (LM) for two values of capital



Notes: The movement of the curves as k changes from 1 to 2 is indicated by arrows. Parameter values: $\sigma_1 = 0.75$, $\sigma_2 = 1.25$, $\varepsilon = 1.5$, $\gamma = 0.5$, $\alpha_1 = \alpha_2 = 0.5$, $M_1 = M_2 = 1$, $k = 1$ or 2. These parameter values imply that the elasticities of output with respect to capital, ϵ_1 and ϵ_2, are both 0.5 in the initial equilibrium.

(CC) in reaction to changes in k or to M in both sectors are discussed in detail below, after derivation of the labor mobility condition (LM).

3.2 The labor mobility condition (LM)

This condition is obtained by combining the optimal allocation of output across sectors from a consumption (final sector) and production (factor allocation) point of view. We first solve for the demand functions for the intermediate goods under perfect competition by maximizing output (1) subject to the zero profit condition $p_1Y_1 + p_2Y_2 = PY$. We normalize $P = 1$. This maximization problem yields the inverse demand functions for the intermediate inputs:

$$p_1 = \gamma \left(\frac{Y_1}{Y}\right)^{-\frac{1}{\varepsilon}} \tag{10}$$
$$p_2 = (1-\gamma)\left(\frac{Y_2}{Y}\right)^{-\frac{1}{\varepsilon}}$$

These two equations yield the relative demand, Y_1/Y_2, as a function of the relative price, p_1/p_2.

$$\frac{p_1}{p_2} = \frac{\gamma}{1-\gamma}\left(\frac{Y_1}{Y_2}\right)^{-\frac{1}{\varepsilon}} \tag{11}$$

Combining this with the condition for the optimal allocation of labor across sectors (7), which requires that $p_1 MPL_1 = p_2 MPL_2$, yields the labor mobility condition

$$LM(\kappa, \lambda, k, M_1, M_2) \equiv \frac{p_1}{p_2}\frac{MPL_1}{MPL_2} = \underbrace{\frac{\gamma}{(1-\gamma)}\left(\frac{Y_1}{Y_2}\right)^{-\frac{1}{\varepsilon}}}_{\text{Relative Price}} \underbrace{\frac{1-\alpha_1}{1-\alpha_2}\frac{(Y_1/\lambda)^{\frac{1}{\sigma_1}}}{(Y_2/(1-\lambda))^{\frac{1}{\sigma_2}}}\frac{M_1^{\frac{\sigma_1-1}{\sigma_1}}}{M_2^{\frac{\sigma_2-1}{\sigma_2}}}}_{\text{Relative Marginal Product}} = 1. \tag{LM}$$

Taking κ as given, this condition describes the optimal allocation of labor across sectors in order to balance two considerations: changing the allocation of labor to a sector affects its relative marginal product, but also its relative price. (Note that λ appears within the condition both explicitly and within the sectoral output terms Y_1 and Y_2.)

For the analysis below, it is useful to consider relative price changes and relative marginal product changes in this equation separately, as they are conceptually distinct. This is also clear from the elasticities of (LM) with respect to key endogenous and exogenous variables:

$$\epsilon_{LM,\lambda} = \epsilon_{\frac{P_1}{P_2},\lambda} + \epsilon_{\frac{MPL_1}{MPL_2},\lambda} = -\frac{1}{\varepsilon}\left((1-\epsilon_1) + (1-\epsilon_2)\frac{\lambda}{1-\lambda}\right) - \left(\frac{1}{\sigma_1}\epsilon_1 + \frac{1}{\sigma_2}\epsilon_2\frac{\lambda}{1-\lambda}\right) \tag{12a}$$

$$\epsilon_{LM,\kappa} = \epsilon_{\frac{P_1}{P_2},\kappa} + \epsilon_{\frac{MPL_1}{MPL_2},\kappa} = -\frac{1}{\varepsilon}\left(\epsilon_1 + \epsilon_2\frac{\kappa}{1-\kappa}\right) + \left(\frac{1}{\sigma_1}\epsilon_1 + \frac{1}{\sigma_2}\epsilon_2\frac{\kappa}{1-\kappa}\right) \tag{12b}$$

$$\epsilon_{LM,k} = \epsilon_{\frac{P_1}{P_2},k} + \epsilon_{\frac{MPL_1}{MPL_2},k} = -\frac{1}{\varepsilon}(\epsilon_1 - \epsilon_2) + \left(\frac{1}{\sigma_1}\epsilon_1 - \frac{1}{\sigma_2}\epsilon_2\right) \tag{12c}$$

$$\epsilon_{LM,M_1} = \epsilon_{\frac{P_1}{P_2},M_1} + \epsilon_{\frac{MPL_1}{MPL_2},M_1} = -\frac{1}{\varepsilon}(1-\epsilon_1) + \left(\frac{\sigma_1-\epsilon_1}{\sigma_1}\right) \tag{12d}$$

$$\epsilon_{LM,M_2} = \epsilon_{\frac{P_1}{P_2},M_2} + \epsilon_{\frac{MPL_1}{MPL_2},M_2} = \frac{1}{\varepsilon}(1-\epsilon_2) - \left(\frac{\sigma_2-\epsilon_2}{\sigma_2}\right) \tag{12e}$$

which all contain separate terms for the relative price effect and the relative marginal product effect. In these expressions, ϵ_s denotes the elasticity of Y_s with respect to K_s.

It is clear from (LM) that for any κ, allocating the entire labor endowment to one of the two sectors is not a solution. Hence, optimal λ is strictly between 0 and 1. This implies that depicted in κ, λ-space, the (LM) curve crosses the (CC) curve strictly once from above.[12]

Moreover, if $\sigma_1, \sigma_2 < \varepsilon$, $\epsilon_{LM,\kappa} > 0$, implying that (LM) is strictly upward-sloping in κ, λ-space. The opposite is the case if $\sigma_1, \sigma_2 > \varepsilon$. These slopes are driven by the balance of the relative price and relative marginal product effects. Consider an increase in κ. This increases both the output and the marginal product of labor in sector 1. Increased output results in a lower relative price in this sector, calling for labor to be reallocated to sector 2. The strength of this effect is driven by the demand elasticity ε. We refer to this as the *relative price effect.* Higher marginal product of labor calls for more labor to be allocated to sector 1. We refer to this as the *relative marginal product effect.* The strength of this effect depends on the degree of complementarity of capital and labor in sector 1 relative to sector 2, and thus on σ_1 and σ_2. If both σ_1 and σ_2 are smaller than ε, the relative marginal product effect dominates, and allocating more capital to a sector also calls for allocating more labor to the sector. Put differently, when $\varepsilon > \sigma_1, \sigma_2$, it is easier to substitute in the final sector than in any of the intermediate good sectors, and therefore the sectoral fractions of capital and labor move in the same direction. In our framework, this is expressed as an upward-sloping (LM) curve. In the opposite case, $\varepsilon < \sigma_1, \sigma_2$, it is easier to substitute in the intermediate sectors that in the final sector. In this case, the relative price effect dominates the relative marginal product effect and an increase in κ requires a decrease in λ: the (LM) curve is downward-sloping.

3.3 Development and structural change

Development, i.e. changes in k and M, shifts the contract curve and the labor mobility condition, leading to changes in the optimal allocation of capital and labor across sectors. Propositions 1 and 2 summarize what structural change looks like in this general model, as a function of parameters and sectoral output elasticities.[13] After stating these results,

[12] Note that there is only one intersection between the two curves since the objective function $F(Y_1(\kappa K, \lambda L), Y_2((1-\kappa)K, (1-\lambda)L)$ is strictly quasi-concave in (λ, κ) and the FOCs of maximization of a strictly quasi-concave function over a convex set yield a unique global maximum; see Takayama (1985, Theorem 1.E.3, p. 115). The strict quasiconcavity of F with respect to λ, κ results from the strict quasi-concavity of F with respect to Y_1 and Y_2 and the strict quasiconcavity of Y_1 and Y_2 with respect to their inputs.

[13] With a CES production function, ϵ_s is not a structural parameter. Each ϵ_s depends on σ_s, α_s, and also on the input allocation in the sector. In spite of this, it is still very useful to define conditions in terms of these elasticities. This is because (7) implies that ϵ_s equals the capital income share in sector s. Given the scant variation and easy observability of factor income shares, their order of magnitude and plausible variation across sectors is in fact easier to assess than that of the structural parameters in this and many

we discuss intuition in terms of the three effects at work here: the factor rebalancing effect that moves (CC), and the relative price and relative marginal product effects moving (LM). Derivations can be found in Appendix A, and Figure 1 is helpful for visualizing the analysis. We then discuss which cases are the most likely ones in practice, relate our results to the literature by analyzing some special cases, and present some simulations that illustrate the theoretical results.

Lemma 1 The factor rebalancing effect. *Assume $\sigma_2 > \sigma_1$. Then, an increase in k shifts the contract curve up in κ, λ-space. A proportional increase in both M_1 and M_2 shifts it down.*

Proof: The result follows immediately from equations (9c), (9d) and (9e).

This result is very intuitive: an increase in k or in M_1 and M_2 corresponds to a shift in the relative abundance of the two inputs, capital and effective labor. This affects relative marginal products and the optimal factor mix in each sector. For instance, for any κ, an increase in k increases the marginal product of labor more in the less flexible sector. Hence, as k increases, (CC) requires larger λ at any κ. The Lemma thus states that the more flexible sector tends to increase its use of the input that becomes more abundant, and reduce its use of the one that becomes relatively more scarce. We call this the *factor rebalancing effect.* The presence of this effect crucially depends on differences in substitution elasticities. As seen above, if $\sigma_1 = \sigma_2$, (CC) is not affected by changes in k, or by proportional changes in M_1 and M_2. The effect is only operational if the sectors differ in their ease of input substitutability.

Putting together these shifts in (CC) with those in (LM) allows us to characterize how development leads to shifts in factors across sectors. All results are collected in Propositions 1 and 2.

First, consider how the optimal allocation changes with k.

Proposition 1 Increases in k. *Assume $\sigma_2 > \sigma_1$. Define the following three conditions:*

$$\left(\frac{1}{\sigma_1} - \frac{1}{\varepsilon}\right)\epsilon_1 > \left(\frac{1}{\sigma_2} - \frac{1}{\varepsilon}\right)\epsilon_2 \tag{P1.A}$$

$$\epsilon_1 - \epsilon_2 < \frac{\sigma_2 - \sigma_1}{\varepsilon - \sigma_2}(1 - \epsilon_1) \tag{P1.B}$$

$$\epsilon_1 - \epsilon_2 > \frac{\sigma_2 - \sigma_1}{\sigma_2 - \varepsilon}\epsilon_1 \tag{P1.C}$$

other models.

Then an increase in the aggregate capital-labor ratio k yields the following changes in λ and κ:

1. *If $\sigma_1, \sigma_2 < \varepsilon$, the relative marginal product effect dominates the relative price effect, and (LM) is upward-sloping in κ, λ-space.*
   (a) *If condition P1.A holds, an increase in k shifts (LM) up.*
      i. *If condition P1.B holds, the upward shift of (CC) at unchanged κ exceeds that of (LM). ($g(\lambda_{CC}) > g(\lambda_{LM})$.) Hence, λ increases and κ decreases.*
      ii. *If condition P1.B does not hold, the upward shift of (LM) at unchanged κ exceeds that of (CC). ($g(\lambda_{LM}) > g(\lambda_{CC})$.) Hence, λ and κ both increase.*
   (b) *If condition P1.A does not hold, an increase in k shifts (LM) down in κ, λ-space. Hence, λ and κ both decrease.*
2. *If $\sigma_1, \sigma_2 > \varepsilon$, the relative price effect dominates the relative marginal product effect, and (LM) is downward-sloping in κ, λ-space.*
   (a) *If condition P1.A holds, an increase in k shifts (LM) up.*
      i. *If condition P1.B does not hold, the upward shift of (CC) at unchanged κ exceeds that of (LM). ($g(\lambda_{CC}) > g(\lambda_{LM})$.) Hence, λ increases and κ decreases.*
      ii. *If condition P1.B holds, the upward shift of (LM) at unchanged κ exceeds that of (CC). ($g(\lambda_{LM}) > g(\lambda_{CC})$.) Hence, λ and κ both increase.*
   (b) *If condition P1.A does not hold, an increase in k shifts (LM) down in κ, λ-space.*
      i. *If condition P1.C holds, the leftward shift of (CC) at unchanged λ exceeds that of (LM). ($|g(\kappa_{CC})| > |g(\kappa_{LM})|$.) Hence, λ increases and κ decreases.*
      ii. *If condition P1.C does not hold, the leftward shift of (LM) at unchanged λ exceeds that of (CC). ($|g(\kappa_{LM})| > |g(\kappa_{CC})|$.) Hence, λ and κ both decrease.*

Proof. See Appendix A. ■

The conditions in Proposition 1 have intuitive economic interpretations. First of all, for $\sigma_2 > \sigma_1$, an increase in k causes an upward or leftward shift in (CC) in κ, λ-space. The reason is that at any level of λ, the more flexible sector is better placed to make use of the now more abundant factor – capital – and changes its input mix to become more capital-intensive. Hence, λ increases or κ declines. This is the *factor rebalancing effect.* Clearly, the shift is larger the larger the difference between the substitution elasticities.

Both the direction and the shift in the (LM) curve depend on parameters and on sectoral output elasticities. First, if σ_1 and σ_2 are smaller than ε, the *relative marginal product effect* dominates the *relative price effect.* Which sector is favored by the relative marginal product effect depends on the optimal response of the labor input to increased capital. Recall that with a CES production function, the marginal product of labor in sector i is proportional to $(Y_i/L_i)^{1/\sigma_i}$. An increased capital input affects this term in two ways. First, for any increase in Y_i, the marginal product of labor increases more the lower σ_i, as this implies stronger complementarity between capital and labor. Hence, the relative marginal product effect favors the less flexible sector. Second, the importance of capital as an input in the sector matters. In the conditions above, this is captured via the elasticities of sectoral output with respect to capital, ϵ_1 and ϵ_2. This reflects that given any σ, the optimal labor input increases more the more output responds to the increase in capital. This is clear from the expression for the marginal product in the CES case. A particularly stark illustration is provided by the Leontief production function $Y_s = \min(\alpha_s K_s, (1-\alpha_s)L_s)$. Here, optimality requires $L_s = \frac{\alpha_s}{1-\alpha_s}K_s$: the optimal increase in the labor input following an increase in K_s increases in the weight on capital, α_s. Hence, the relative marginal product effect favors the sector with the lower substitution elasticity, and the sector with the higher elasticity of output with respect to capital.

The direction of shift of (LM) then depends on both the substitution elasticities and their relation to the demand elasticity, and on how much the output of each sector increases in response to a larger capital input. If for example $\epsilon_1 = \epsilon_2$, condition P1.A holds, and (LM) shifts up. This case illustrates how the relative marginal product effect favors the less flexible sector. Similarly, if $\sigma_1 = \sigma_2 < \varepsilon$ but $\epsilon_1 > \epsilon_2$, condition P1.A holds, and (LM) shifts up. This case illustrates how the relative marginal product effect favors the sector with higher ϵ_s. Clearly, (LM) also shifts up if $\sigma_2 > \sigma_1$ and $\epsilon_1 > \epsilon_2$. Only if $\sigma_2 > \sigma_1$ and $\epsilon_1 < \epsilon_2$ do the two forces work in opposite directions. Then, if the difference in ϵ is sufficiently large or that in σ is sufficiently small, (LM) can shift down. Condition P1.A determines whether this is the case.

When both (CC) and (LM) shift up, it is clear that λ must increase, as both conditions call for more labor to be allocated to sector 1. What happens with capital depends on the relative size of the two shifts. Figure 1 illustrates the case in which (CC) shifts up more, and κ declines: capital and labor move in opposite directions (Case 1.a.i.). Consider how much λ needs to change for each condition to hold, supposing that κ does not change. Denote these changes, which can be computed using the elasticities in equations (9a), (9c), (12a) and

(12c), by $g(\lambda_{CC})$ and $g(\lambda_{LM})$ respectively. Comparing them leads to condition P1.B. The economic interpretation of this condition is, again, intuitive. First, note that while $\sigma_2 > \sigma_1$ tends to shift both (CC) and (LM) up, the effect on (CC) is stronger. This is because the strength of the factor rebalancing effect depends directly on the difference between σ_2 and σ_1, while the relative marginal product effect depends on the product of the σ's and the ϵ's, which are smaller than one. Second, the shift in (CC) is independent of the ϵ's, while (LM) shifts up more the larger $\epsilon_1 - \epsilon_2$. Hence, the condition states that (LM) shifts up less than (CC) if $\epsilon_1 - \epsilon_2$ is not too large, where the size of the threshold depends on the difference between the σ's. As this affects (CC) more strongly, a larger difference between the σ's implies that a larger difference $\epsilon_1 - \epsilon_2$ is required to shift (LM) more than (CC). Economically speaking, the condition ensures that the factor rebalancing effect dominates the relative marginal product effect.

As a result, it is clear from Proposition 1 that depending on which effect dominates, different outcomes in terms of sectoral reallocations will occur. Whenever the factor rebalancing effect dominates, the allocation of capital to sector 1, the less flexible sector, declines. In addition, in case 1.a. and 2.a., the allocation of labor to sector 1 increases if the relative marginal product effect dominates the relative price effect, and ϵ_1 is not too small relative to ϵ_2. The allocation of both labor and capital to sector 1 increases only if the relative marginal product effect is so strong (because of large ϵ_1 relative to ϵ_2) that it also outstrips the factor rebalancing effect.

Two important remarks follows.

Remark 1 to Proposition 1. In all cases in Proposition 1 where λ increases and κ decreases in response to an increase in the aggregate capital-labor ratio, both sectoral capital-labor ratios, $k_1 \equiv K_1/L_1$ and $k_2 \equiv K_2/L_2$, also increase. (See Appendix A for a proof.)

Hence, following an increase in the aggregate capital-labor ratio, we should not expect to see declines in the *level* of the capital-labor ratio in any sector. What the proposition predicts are different, positive, growth rates.

Remark 2 to Proposition 1. There is a level of k (let it be $\bar{k}$) such that $\kappa = \lambda$.[14] At this point, a *factor intensity reversal* occurs: With $\sigma_2 > \sigma_1$, $\kappa > \lambda$ for $k < \bar{k}$, and $\kappa < \lambda$ for $k > \bar{k}$.

An allocation with $\kappa = \lambda$ implies that the aggregate capital labor ratio and those in the two sectors are all equated. This is only possible if the contract curve coincides with the

[14] It is clear from (CC) that this level is such that $\bar{k}^{\frac{1}{\sigma_1} - \frac{1}{\sigma_2}} = \frac{\alpha_1}{\alpha_2} \frac{1-\alpha_2}{1-\alpha_1} M_1^{\frac{1-\sigma_1}{\sigma_1}} M_2^{\frac{\sigma_2-1}{\sigma_2}}$.

45-degree line. (CC) defines the level of k for which this occurs. Given $\sigma_2 > \sigma_1$, an increase in k above this level will then shift the contract curve up in κ, λ-space (Lemma 1), implying that for $k > \bar{k}$, it must be that $\lambda > \kappa$ at the optimal allocation. The reverse argument applies for $k < \bar{k}$.

Intuitively, above $\bar{k}$, capital is relatively abundant, and its rental price relatively low. The more flexible sector is freer to respond to this, implying that it uses it more intensively, so that $\kappa < \lambda$ in this range. Below $\bar{k}$, capital is relatively scarce, so its rental price is relatively high. The more flexible sector can more easily substitute to labor, implying $\kappa > \lambda$ in this range. One could thus say that at $\bar{k}$, the economy's endowments of capital and labor are "balanced" given the production technology of sector 1.

Next, consider how the optimal allocation changes with a proportional change in M_1 and M_2.

Proposition 2 Increases in M_1 and M_2. *Assume $\sigma_2 > \sigma_1$. Then a proportional increase in both M_1 and M_2 leads to the following changes in λ and κ:*

1. *$\sigma_1, \sigma_2 < \varepsilon$. As shown above, the relative marginal product effect dominates the relative price effect, and (LM) is upward-sloping in κ, λ-space in this case.*

   (a) *If condition P1.A holds, a proportional increase in M_1 and M_2 shifts (LM) down in κ, λ-space.*

      i. *If condition P1.B holds, the downward shift of (CC) at unchanged κ exceeds that of (LM). ($|g(\lambda_{CC})| > |g(\lambda_{LM})|$.) Hence, λ declines, but κ increases.*

      ii. *If condition P1.B does not hold, the downward shift of (LM) at unchanged κ exceeds that of (CC). ($|g(\lambda_{CC})| < |g(\lambda_{LM})|$.) Hence, both λ and κ decrease.*

   (b) *If condition P1.A does not hold, a proportional increase in M_1 and M_2 shifts (LM) up. Given the downward shift in (CC) and the positive slope of (LM), this implies that λ and κ both increase.*

2. *$\sigma_1, \sigma_2 > \varepsilon$. As shown above, the relative price effect dominates the relative marginal product effect, and (LM) is downward-sloping in this case.*

   (a) *If condition P1.A holds, a proportional increase in M_1 and M_2 shifts (LM) down in κ, λ-space.*

      i. *If condition P1.B holds, the downward shift of (LM) at unchanged κ exceeds that of (CC). ($|g(\lambda_{CC})| < |g(\lambda_{LM})|$.) Hence, both λ and κ decline.*

ii. *If condition P1.B does not hold, the downward shift of (CC) at unchanged κ exceeds that of (LM). ($|g(\lambda_{LM})| < |g(\lambda_{CC})|$.) Hence, λ declines, but κ increases.*

(b) *If condition P1.A does not hold, a proportional increase in M_1 and M_2 shifts (LM) up.*

i. *If condition P1.C holds, the rightward shift of (CC) at unchanged λ exceeds that of (LM). ($g(\kappa_{CC}) > g(\kappa_{LM})$.) Hence, κ increases and λ declines.*

ii. *If condition P1.C does not hold, the rightward shift of (LM) at unchanged λ exceeds that of (CC). ($g(\kappa_{LM}) > \mathrm{d}\kappa_{CC}$.) Hence, λ and κ both increase.*

Proof. See Appendix A. ■

Again, the conditions are economically intuitive. For an increase in M_1 and M_2, the intuition for the change in the contract curve is the same as above: the more flexible sector substitutes towards the input that becomes more abundant. The intuition for the change in the labor mobility condition is slightly different: Given a change in labor-augmenting productivity, the marginal product of labor increases more in the more flexible sector, given the lower curvature of the production function there. The intuition regarding output elasticities is unchanged. Hence, condition P1.A now implies that if the difference in substitution elasticities is large relative to the difference in output elasticities, the relative marginal product effect favors the more flexible sector. This calls for resources to move out of the less flexible sector. This can be represented as a downward shift in (LM). Similarly, condition P1.B states that if output elasticities are similar, the factor rebalancing effect dominates the relative marginal product effect. In this case, while labor still leaves sector 1 due to the rebalancing effect, some capital flows in to compensate.

Finally, we can characterize the behavior of the relative price, relative output, and relative value added of the two sectors.

Proposition 3 *The relative price p_1/p_2 declines in k for $k < \bar{k}$, is minimized at $\bar{k}$, and increases for $k > \bar{k}$. Relative output Y_1/Y_2 increases in k for $k < \bar{k}$, is maximized at $\bar{k}$, and decreases for $k > \bar{k}$. Relative value added $p_1Y_1/(p_2Y_2)$ declines (increases) in k for $k < \bar{k}$, is minimized (maximized) at $\bar{k}$, and increases (declines) for $k > \bar{k}$ if $\varepsilon < (>)1$. It is constant at $\gamma/(1-\gamma)$ if $\varepsilon = 1$.*

Proof. See Appendix A. ■

At $k = \bar{k}$, the relative price of sector 1 output is minimized, because its relative marginal cost is minimized at this point. From the discussion of Remark 2 to Proposition 1, when k is larger or smaller than $\bar{k}$, k is relatively abundant or scarce. This imbalance affects the marginal cost of the less flexible sector more, implying that p_1/p_2 increases as k moves away from $\bar{k}$. The consequence is familiar: if the products of the two sectors are gross complements (substitutes), i.e. if $\varepsilon < (>)1$, the value added share of the less flexible sector increases (decreases) as k moves away from $\bar{k}$. Hence, the model predicts that changes in k generate structural change in terms of both inputs and value added.

3.4 Plausible parameter ranges, and three special cases

Propositions 1 and 2 feature several conditions on parameters and sectoral output elasticities. Which ones are plausible to hold, and as a consequence, which cases are most realistic? To answer this question, information on sectoral factor income shares, substitution elasticities, and the substitution elasticity in the production of the final good are required.

Among these, the most reliable information is that on factor income shares. Valentinyi and Herrendorf (2008) compute factor income shares for broad sectors defined in several different ways. The highest capital income share they report is that for agriculture, at 0.54, due to a high share of land (included in the capital share in their calculations). For other sectors, the capital income share lies between 0.28 and 0.4. Hence, suppose that except for agriculture, the difference between factor income shares is at most 0.1. What restrictions on the other parameters does this imply?

It turns out that with a small difference in ϵ, condition P1.A holds in many cases. Consider for instance a situation with $\epsilon_1 = 0.3$, $\epsilon_2 = 0.4$, and $\sigma_1 = 0.75$; a value in line with many estimates for the aggregate economy. (See Leon-Ledesma, McAdam and Willman (2010) for an overview of estimates.) Then, for ε below 0.75, condition P1.A holds for any $\sigma_2 > \sigma_1$, and for ε between 0.75 and a very high value of 1.5, the condition holds for any σ_2 larger than 0.86 – only slightly larger than σ_1. Condition P1.A may thus plausibly hold in reality, as it only requires small differences between sectoral substitution elasticities, given our knowledge of factor income shares. This implies that the factor rebalancing effect may dominate for plausible parameter values.

Thus, suppose that condition P1.A holds, and let us move on to the other conditions. Given our assumptions, condition P1.B never holds when ε is below 0.75, whereas it always holds when $\varepsilon > 0.75$. Hence, cases 2.a.i. and 1.a.i. of Proposition 1 and cases 2.a.ii. and 1.a.i. of Proposition 2, in which λ and κ move in opposite directions following an increase in k or productivity, are quite likely to occur empirically.

It is also useful to relate the discussion here to two prominent cases in the literature, and to discuss a special case that is important in the analysis of dynamics.

3.4.1 Special case 1: Optimal allocation in the simplified Ngai-Pissarides model

In NP, $\sigma_1 = \sigma_2 = 1$ and $\alpha_1 = \alpha_2$, so that $\epsilon_1 = \epsilon_2$. ε can be either larger or smaller than 1, and hence larger or smaller than $\sigma_{1,2}$. As seen above, these parameter values imply that (CC) does not react to changes in k or to proportional changes in M_1 and M_2. Secondly, $\epsilon_1 = \epsilon_2$ combined with $\sigma_1 = \sigma_2 = 1$ implies that (LM) does not shift either in response to changes in k or to proportional changes in M_1 and M_2. This is because $\epsilon_1 = \epsilon_2$ implies that the relative price effect is not operative for changes in k and in M. Nor is the relative marginal product effect if both $\epsilon_1 = \epsilon_2$ and $\sigma_1 = \sigma_2$.

(LM) does, however, shift in response to changes in the ratio M_1/M_2. An increase in M_1, for example, shifts (LM) up if $\sigma_1 = \sigma_2 < \varepsilon$. Since the factor rebalancing effect is shut down in this scenario, structural change only depends on the relative marginal product effect versus the relative price effect. For a change in the ratio M_1/M_2, the relative marginal product effect dominates if $\sigma < \varepsilon$, or if $1 < \varepsilon$ in the case where both sectors' production functions are Cobb-Douglas.[15]

3.4.2 Special case 2: Optimal allocation in the Acemoglu-Guerrieri model

In AG, $\sigma_1 = \sigma_2 = 1$ and $\alpha_1 \neq \alpha_2$, so that $\epsilon_1 \neq \epsilon_2$. ε can be larger or smaller than 1. In this case, if $\varepsilon > 1$, the relative marginal product effect dominates the relative price effect. As a result, an increase in k shifts (LM) up if $\alpha_1 > \alpha_2$ and thus leads to reallocation of both factors towards the more capital-intensive sector 1. The reverse occurs if $\varepsilon < 1$. This is, of course, in line with the results reported in AG.

3.4.3 A tractable special case of the model of unequal sectoral capital-labor substitution

Finally, in the dynamic analysis of the model of unequal sectoral substitution possibilities in Section 4.3, considerations of tractability force us to focus attention on the case where $\varepsilon = 1$ and one sector is Cobb-Douglas. Is this restrictive? Consider first the case where one sector

[15] Note that there is a slight difference between the treatment in NP and the discussion here, as NP model neutral and not labor-augmenting technical progress. With CES production, this makes a difference. As a result, the key threshold value for ε here is σ, not 1, generalizing the result in NP. For the rest, the key forces at work are the same.

is more flexible than Cobb-Douglas, i.e. $\varepsilon = \sigma_1 = 1 < \sigma_2$. Here, $\sigma_1 = \varepsilon$ implies that condition P1.A reduces to $\sigma_2 > \sigma_1$, and therefore holds in all cases of interest. Inspecting (LM), $\sigma_1 = \varepsilon$ implies that the denominator of (LM) does not change with k. $\sigma_2 > \varepsilon$ then implies that no matter the values of ϵ_1 and ϵ_2, the denominator of (LM) decreases in k. This implies an upward shift of (LM), as higher λ is required to compensate the effect of higher k, at any κ. In other words, the relative price effect weakly dominates the relative marginal product effect in this parameter configuration. The same is true in the case of $\sigma_1 < 1 = \sigma_2 = \varepsilon$. This is similar to other analyses of structural change in the literature, like AG and NP, where the relative price effect plays a powerful role.

What is different here is the presence of the factor rebalancing effect. Returning to the case where $\sigma_1 = \varepsilon = 1 < \sigma_2$, $\sigma_1 = \varepsilon$ implies that the right hand side of P1.B becomes $\epsilon_1 - 1$, so that P1.B cannot hold unless $\epsilon_2 = 1$. As a result, λ increases and κ decreases as k increases, and λ declines and κ increases as both M_1 and M_2 increase. (Cases 2.a.i of Proposition 1 and 2.a.ii of Proposition 2.) In other words, in both cases, the factor rebalancing effect dominates the relative price effect (minus the relative marginal product effect).

Now consider the case where one sector is less flexible than Cobb-Douglas, i.e. $\sigma_1 < 1 = \varepsilon = \sigma_2$. In this case, the right hand side of P1.B goes to infinity , so that the condition always holds. The equality $\sigma_2 = \varepsilon$ thus implies that the relative price effect can never dominate the factor rebalancing effect. As a consequence, in this scenario, λ increases and κ decreases in k already for relatively small differences in σ (case 1.a.i of Proposition 1), while the opposite occurs for proportional increases in M_1 and M_2 (case 1.a.ii of Proposition 2).

For later use, summarize this result in

Proposition 4 *Assume $\varepsilon = 1$, $\alpha_2 = \alpha_1 = \alpha$ and $\sigma_2 > \sigma_1 = 1$. Then the fraction of capital allocated to the less flexible sector falls as the economy's aggregate capital-labor ratio increases, while the fraction of labor in the same sector increases. Similarly, the fraction of capital (labor) allocated to the less flexible sector does not change when its level of TFP increases but it increases (decreases) with the level of TFP in the more flexible sector and therefore it increases (decreases) with a proportional increase in the level of labor-augmenting* *productivity in the two sectors.*[16] *More specifically, we obtain*

$$\frac{\partial \kappa}{\partial k} = \frac{(1-\sigma_2)}{\sigma_2 G(\kappa) k} < 0 \quad (13a)$$

$$\frac{\partial \lambda}{\partial k} = \left(\frac{\alpha}{1-\alpha}\right)\left(\frac{\lambda(\kappa)}{\kappa}\right)^2 \frac{(\sigma_2-1)}{\sigma_2 G(\kappa) k} > 0 \quad (13b)$$

$$\frac{\partial \kappa}{\partial M_2} = \frac{(\sigma_2-1)}{\sigma_2 G(\kappa) M_2} > 0 \quad (13c)$$

$$\frac{\partial \lambda}{\partial M_2} = \left(\frac{\alpha}{1-\alpha}\right)\left(\frac{\lambda(\kappa)}{\kappa}\right)^2 \frac{(1-\sigma_2)}{\sigma_2 G(\kappa) M_2} < 0 \quad (13d)$$

$$\frac{\partial \kappa}{\partial M_1} = \frac{\partial \lambda}{\partial M_1} = 0, \quad (13e)$$

where

$$G(\kappa) \equiv \left[\frac{1}{\sigma_2\left(1-\lambda(\kappa)\right)} + \frac{1}{\lambda(\kappa)}\right]\left(\frac{\lambda(\kappa)}{\kappa}\right)^2\left(\frac{\alpha}{1-\alpha}\right) + \left[\frac{1}{\kappa} + \frac{1}{\sigma_2(1-\kappa)}\right]$$

and $\lambda(\kappa)$ *is given by (CC). The inequality signs in (13a)-(13d) are reversed when sector 1 is the more flexible one, i.e.* $\sigma_2 < \sigma_1 = 1$. *See Appendix A for a detailed derivation.*

These patterns of sectoral reallocations contrast with those obtained in the NP and AG models, where a sector was either increasing its share of both capital and labor, or decreasing both. The reason is that in these models, sectoral reallocations depend only on the balance between the relative marginal product effect and the relative price effect. As a consequence, the fractions of capital and labor used in any of the two sectors move in the same direction. Put in terms of Figure 1, changes in the state variables only shift the (LM) curve, tracing out points along the (CC) curve. In contrast, in the model of unequal capital-labor substitution, sectoral reallocations are additionally driven by the factor rebalancing effect, i.e. by changes in the relative input price, the ratio of the wage per unit of effective labor to the rental rate. Cross-sectoral differences in the elasticity of substitution then allow the more flexible sector to absorb a larger fraction of the relatively cheap input and to release some of the relatively expensive one. In this case, changes in state variables not only shift the (LM) curve, but also shift the (CC) curve. The latter implies that there can be a negative relationship between the fractions of capital and labor within each sector.[17]

[16] So far we have characterized the response of sectoral allocations to proportional changes in sectoral TFP's. As in AG and NP, in this special case we are able to characterize the response of input allocations to changes in the levels of TFP for either of the two or for both sectors.

[17] Finally, and again in contrast to the NP and AG models, in the unequal capital-substitution model sectoral reallocations take place despite the fact that the production function for final output is Cobb-Douglas, i.e. sectoral reallocations take place even when the expenditure shares of the intermediate goods remain constant.

Summarizing, the true restriction implicit in the parametric assumptions in Proposition 4 is that the factor rebalancing effect dominates the other effects. This implies that our analysis in Section 4 focuses on a scenario where our new channel of interest – factor rebalancing – is important. The theoretical analysis above has identified under which conditions this is or is not the case, and has illustrated that this can occur for a broad range of parameter combinations.

Next, we present some simulations that illustrate the analysis so far and also allow us to extend results to the case with $\sigma_1 < \varepsilon < \sigma_2$. In addition, they contain results on the evolution of the relative price and sectoral value added shares as k or M change.

3.5 A numerical illustration

The goals of this analysis are to illustrate the workings of the model and to assess the robustness of our analytical results to changes in parameter configurations, rather than to reproduce observed patterns of structural change. These most likely arise from a combination of several forces, of which differences in the sectoral elasticity of substitution are but one. Our analysis abstracts from these other forces, to illustrate more clearly the role of sectoral differences in substitution possibilities.

Our strategy for choosing parameters is consistent with these goals. In order to abstract from other drivers of structural change and to focus only on the impact of differences in the elasticity of substitution, we set both sectoral levels of TFP equal, $M_1 = M_2 = M$, and set the distributional share of capital to one half for both sectors, $\alpha_s = 0.5$. Under these restrictions the essence of the capital-labor substitution mechanism can be conveniently illustrated by combining equations (7) and (8) to reach,

$$\tilde{k}_1^{\frac{1}{\sigma_1}} = \tilde{k}_2^{\frac{1}{\sigma_2}} = \frac{w}{MR} \equiv \tilde{\omega} \tag{14}$$

where $\tilde{k}_s$ is capital per unit of effective labor in sector s and $\tilde{\omega}$ is the ratio of the wage per unit of effective labor to the rental rate. To be concise, we will refer to it as the relative input price.

In general equilibrium, increases in capital per unit of effective labor, i.e. increases in k or reductions in M, bring about increases in the relative input price. In response to this, the sector with higher elasticity of substitution takes advantage of this change by increasing its level of capital per unit of effective labor faster than the other sector. This is, of course, the factor rebalancing effect. In line with this, it proves useful to track the evolution of key variables in terms of the relative input price, $\tilde{\omega}$. Notice that given $\tilde{\omega}$, we can use (14) to determine sectoral levels of capital per unit of effective labor, $\tilde{k}_s(\tilde{\omega}) = \tilde{\omega}^{\sigma_s}$. These, in turn, allow to calculate the sectoral labor income shares, given by $1 - \epsilon_s(\tilde{\omega}) = \dfrac{\tilde{\omega}}{\tilde{k}_s(\tilde{\omega}) + \tilde{\omega}}$. Then (4) determines the levels of sectoral output per unit of effective labor, $f_s(\tilde{k}_s)$, and marginal products, $f_s'(\tilde{k}_s)$. Combining these results with (7) determines the relative price of intermediate goods, $p(\tilde{\omega}) \equiv \dfrac{p_1}{p_2} = \dfrac{f_2'(\tilde{k}_2)}{f_1'(\tilde{k}_1)}$. Imposing the sectoral market clearing condition that equates relative supply, $p(\tilde{\omega})\dfrac{Y_1}{Y_2} = p(\tilde{\omega})\dfrac{\lambda}{(1-\lambda)}\dfrac{f_2(\tilde{k}_2)}{f_1(\tilde{k}_1)}$, to relative demand, given by (11), we recover the sectoral factor allocations, $\lambda(\tilde{\omega})$ and $\kappa(\tilde{\omega})$. Finally, the aggregate labor income share and capital per unit of effective labor are given by $1 - \epsilon = \lambda(1-\epsilon_1) + (1-\lambda)(1-\epsilon_2)$ and $\tilde{k} = \dfrac{\lambda}{\kappa}\tilde{k}_1$, respectively.

In order to explore the evolution of these variables we need to choose specific values for four parameters: $\gamma, \varepsilon, \sigma_1$ and σ_2. Consistent with our goal, illustration rather than realism, we assume that both intermediate inputs are equally important in the production of final output: $\gamma = 0.5$. In our baseline, we set the sectoral elasticities around the Cobb-Douglas threshold with $\sigma_1 = 0.75$ and $\sigma_2 = 1.25$. These values are within the range of sectoral elasticities estimated by Herrendorf et al (2014). Concerning the elasticity of substitution in the final sector, we will report two sets of results. First, we assume that sectoral outputs are gross complements, and set $\varepsilon = 0.5$. Second, we assume sectoral outputs are gross substitutes, and set $\varepsilon = 1.5$. The first case corresponds to case 2 in Propositions 1 and 2, just like the special case in Proposition 4. The second case corresponds to case 1 in Propositions 1 and 2. Then we explore the case where both sectoral substitution elasticities lie either below or above one. Finally, we also report results for the case where $\varepsilon = 1$. Given the baseline values of σ_1 and σ_2, this case is not covered by Propositions 1 and 2. In this case, the simulations thus complement the analytical results above.[18]

Figure 2 presents the results of these simulations. The left panels illustrate the case of gross complements, while the case of gross substitutes is reproduced in the right panels. Since there is a monotonically increasing relationship between the relative input price and aggregate capital per unit of effective labor, as shown in the top panel, one can think of the x-axis in the last six panels of this figure in terms of the latter variable. As the relative input price increases, the flexible sector is in a better position to take advantage of this

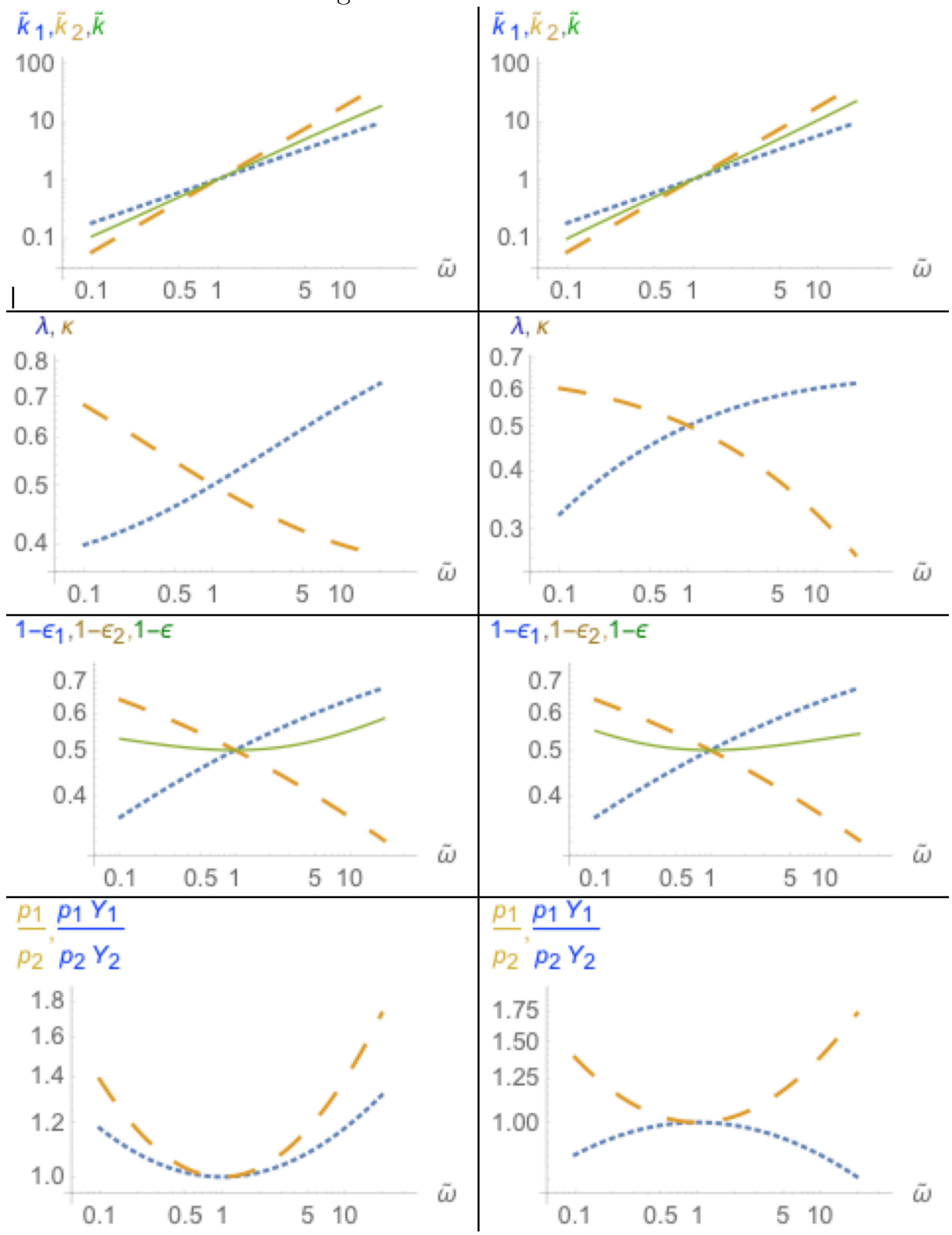
[18] An additional issue concerns the range of $\tilde{\omega}$ we consider in our numerical exercise. We report values in the range $\tilde{\omega} \in [0.1, 20]$. To get a sense of the meaning of this range, consider a Solow model with $s = 0.2, \delta = 0.05$ and $m = n = 0$. Here, if we move from 10% to 200% of the steady state capital stock, the relative input price increases from 2 to 17.5. If the same exercise is conducted assuming $m = 0.02$ and $n = 0.01$, the relative input price for the same range of the steady state capital per unit of effective labor goes from 2.5 to 16.

decrease in the relative price of capital and, therefore, the fraction of capital (labor) that it employs increases (falls). (See the second row of panels.) Notice that when effective labor is relatively abundant, for instance $\tilde{\omega} = 0.5$, the flexible sector employs roughly 54% of the labor force and only 45% of the capital stock. This situation is reversed as effective labor becomes relatively scarce, and thus more expensive. For instance, as $\tilde{\omega}$ increases to 5, the flexible-sector fraction of labor falls by more than 16 percentage points, while its fraction of the capital stock increases by almost 13 percentage points. As a result, the flexible sector is more labor-intensive when the aggregate ratio of capital to units of effective labor is low, i.e. effective labor is abundant, and more capital-intensive when the aggregate ratio of capital to effective labor is high. Given our parameter choices, this capital-intensity reversal takes place when aggregate capital per unit of effective labor is one.

The third panel of Figure 2 reports the evolution of the labor income shares. The relevant benchmark for understanding this evolution is given by the Cobb-Douglas case. With unitary sectoral elasticities of substitution, $\sigma_s = 1$, factor income shares are independent of relative factor prices. Since $\tilde{k}_s^{\frac{1}{\sigma_s}} = \tilde{\omega}$, an increase in the relative factor price leads to a proportional decrease in relative factor use, leaving the factor income share unchanged. (Note that the ratio of the labor to the capital income share equals $\tilde{\omega}/\tilde{k}_s$, which is constant if $\sigma_s = 1$.) If, as in Figure 2, the elasticity of the flexible sector exceeds one, this last expression implies that an increase in the relative input price leads to a more than proportional increase in capital per unit of effective labor. As a result the labor income share in the flexible sector declines as capital per unit of effective labor increases. The reverse is true in the other sector, where the elasticity of substitution is less than one. As a consequence, its labor income share, $1 - \epsilon_1$, increases as capital accumulates. Since the aggregate labor income share is an average, weighted by sectoral labor allocations, of the two labor income shares, its evolution turns out to be mainly determined by that of the labor income share of the labor-intensive sector. When effective labor is relatively abundant, the flexible sector 2 is more labor intensive and therefore the aggregate labor income share decreases as capital accumulates. In contrast, when capital becomes relatively abundant, the less flexible sector becomes labor-intensive and the aggregate labor income share increases with capital accumulation.

The last panel in the left column of Figure 2 presents the evolution of the relative price of the two sectors' output and of the share of value added produced in sector one. Recall that the two are related by the condition for optimal demand from the final goods sector. The path of the relative sectoral price is non-monotonic, with a minimum at the capital-intensity reversal, as indicated by Proposition 3. At this point, the cost advantage enjoyed

Figure 2: Model simulations



Notes: Parameter settings for the simulations: $M_s = L = 1, \gamma = \alpha_s = 1/2, \sigma_1 = 0.75, \sigma_2 = 1.25$, $s = 1, 2$. ε equals 0.5 in the left column and 1.5 in the right column. First variable (dotted line), second variable (dashed line), third variable (solid line). Relative input price $\tilde{\omega} = \omega/(RM)$ on the horizontal axis of each graph. Both axes are on a log scale in each graph.

by the more flexible sector due to its higher elasticity of substitution vanishes, and sectoral levels of capital per unit of effective labor coincide. As capital per unit of effective labor increases above (falls below) one, the flexible sector takes advantage of the change in the relative input price, reducing its unit costs relative to those of sector 1. As a result, the relative sectoral price increases as aggregate capital per unit of effective labor moves away from one, in either direction. Since sectoral outputs are gross complements in the production of final output, changes in the relative sectoral price induce less than proportional changes in relative quantities. As a result, the path of relative value added – a common measure of structural change – is similar to that of the relative sectoral price.

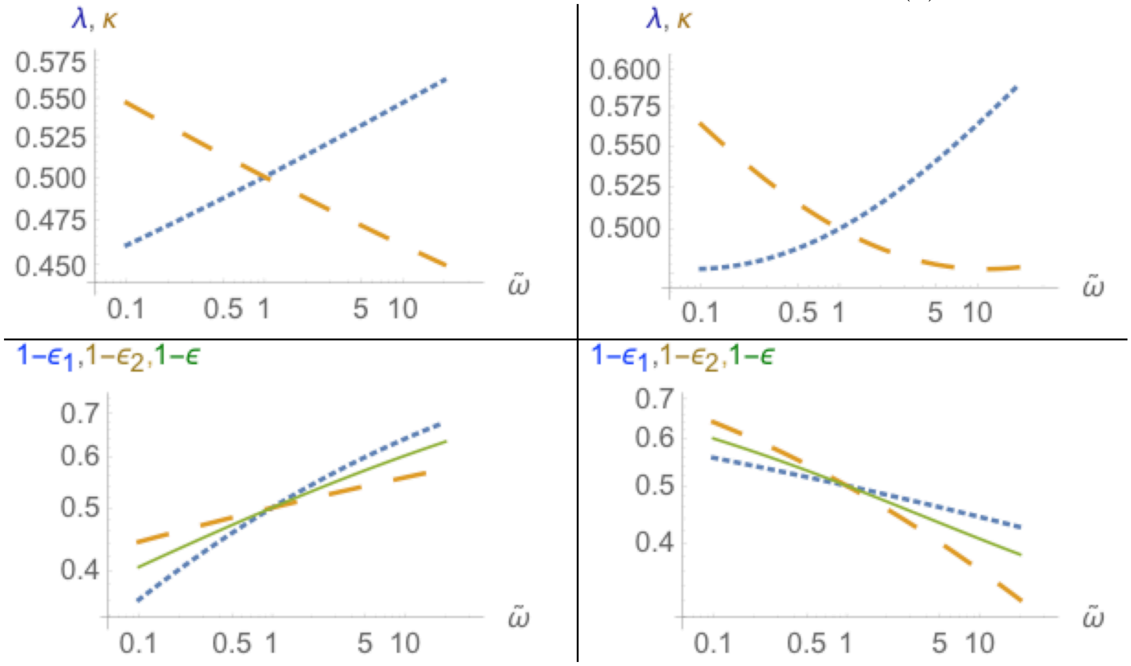
The right panels of Figure 2 illustrate the case where the elastiticity of substitution in the final sector exceeds one. The evolution of the ratios of capital to effective labor, input allocations, and labor income shares is qualitatively similar to that under gross complements. Nonetheless, since in this case sectoral outputs are gross substitutes in the production of final output, relative quantities change more than proportionally than the relative sectoral price. As a result, relative value added moves in the opposite direction of the relative sectoral price.

Figure 3 shows results for the case where both sectoral elasticities lie either below one, $\sigma_1 = 0.75$ and $\sigma_2 = 0.9$, or above one, $\sigma_1 = 1.1$ and $\sigma_2 = 1.25$. The most relevant qualitative difference compared to the previous examples lies in the evolution of the factor income shares. When both sectors are below the Cobb-Douglas threshold, increases in the relative input price are associated with increases in the labor income share in both sectors. They are stronger in the less flexible sector. The converse is true when both sectoral elasticities exceed one.[19]

Finally, we explored cases where the sectoral substitution elasticities lie on both sides of the final output one, $\sigma_1 < \varepsilon < \sigma_2$. The qualitative patterns coincide with those described above and, specifically, in each sector the fractions of capital and labor move in opposite directions as capital per unit of effective labor increases. Of course, since this exercise

[19]Both columns of Figure 3 illustrate parameter configurations that fall in case 2 of Propositions 1 and 2. This example also illustrates that the economy can switch between cases of the Propositions. Specifically, when both sectoral elasticities are above one (right panel) the system begins in case 1.2.b (or 2.2.b.i), for very low $\tilde{\omega}$. Then, when $\tilde{\omega}$ reaches 0.1, it moves to case 1.2.a.i (or 2.2.a.ii) and, when $\tilde{\omega} = 12.5$, it moves to case 1.2.a.ii (or 2.2.a.i). As a result, the paths of factor allocations become non-monotonic, with the fractions of capital and labor in a sector moving in the same direction for very low and very high values of the relative input price and moving in opposite direction for intermediate values of this price. The reason for this is that with CES production functions, output elasticities change with input allocations. For instance, when capital is very scarce (low $\tilde{\omega}$), the elasticity of output with respect to capital in the less flexible sector at the optimal allocation is much higher than that in the more flexible sector, weakening the factor rebalancing effect for any difference in σ. As capital accumulates, the difference in output elasticities is reduced, but the difference in σ remains, so that the factor rebalancing effect gains importance.

Figure 3: Model simulations, different values of σ_s (a)



Notes: Parameter settings as in Figure 2, except that $\sigma_1 = 0.75$ and $\sigma_2 = 0.9$ in the left column and $\sigma_1 = 1.1$ and $\sigma_2 = 1.25$ in the right column, and $\varepsilon = 0.5$ in both columns.

assumes $\varepsilon = 1$, sectoral value added shares remain constant. The results of these simulations are reproduced in Figure 4.

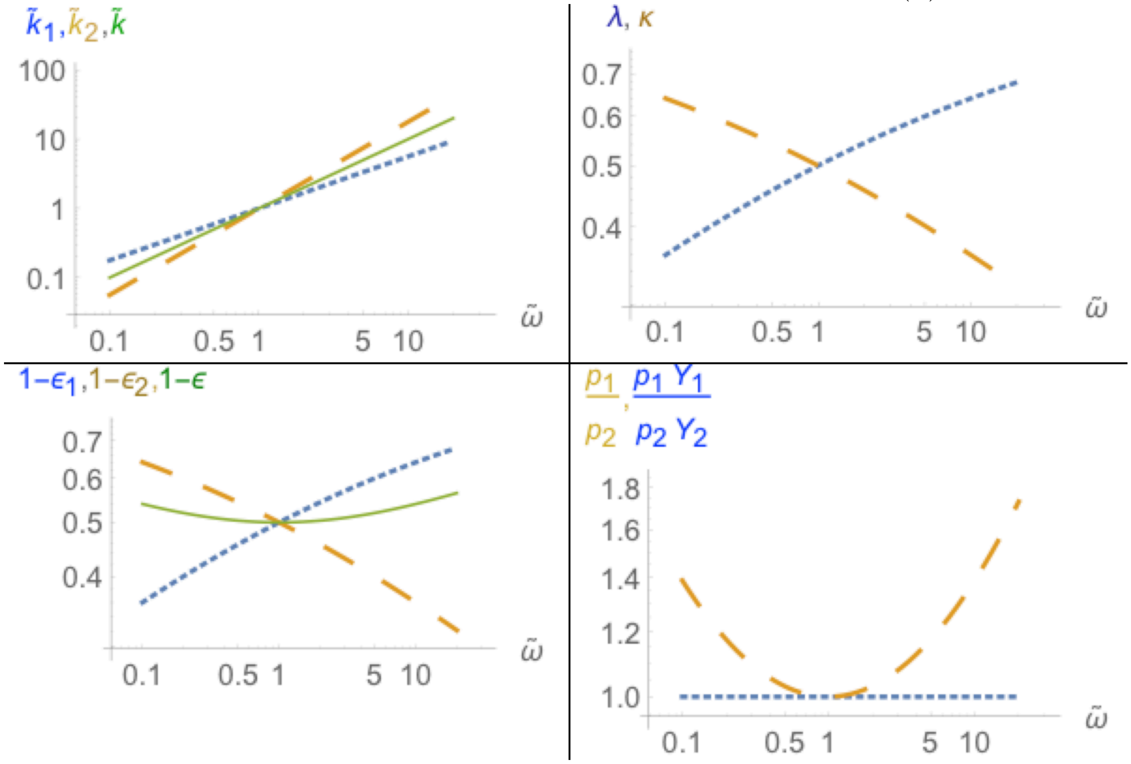
Overall, the patterns emerging from these numerical exercises are exactly in line with Propositions 1 and 2. They clearly illustrate the basic message of these propositions, namely that sectoral differences in the elasticity of substitution can lead to opposite changes in the fractions of labor and capital used in a sector. They also show that in addition, there is structural change in terms of value added.[20]

4 The dynamic problem

The previous section has analyzed structural change for arbitrary, one-time changes in k or M. In this section, we turn to the characterization of the solution for the full dynamic

[20]When interpreting patterns like those in the second row of Figure 2, recall that they refer to shares of capital and labor employed in each sector: actual levels are not predicted to decline. (See Remark 1 to Proposition 1.) Resulting growth rate differences do occur; see for example those cited in the introduction. More generally, they may sometimes be obscured for two reasons. First, our analysis here focusses on clearly illuminating one single mechanism of structural change. In the data, several mechanisms can be at work at the same time. Second, productivity and capital tend to increase together in the data, but they can have opposite implications in terms of structural change, as shown in Propositions 1 and 2. This issue is addressed in the next subsection.

Figure 4: Model simulations, different values of σ_s (b)



Notes: Parameter settings as in Figure 2, except that $\varepsilon = 1$.

problem. For tractability, we are forced to limit the analysis to the three particular cases introduced in Section 3.4, each of which showcases a particular driver of structural change. For added generality, we now also allow the sectoral TFP growth rates to be different in the AG case, and not only in the NP case. Concretely, we assume that $m_2 > m_1 \geq 0$.[21] Also, in line with the case typically stressed in the literature, assume that $\varepsilon < 1$. (Results are reversed but similar when $\varepsilon > 1$.) Under these restrictions, sector 1 is the asymptotically dominant sector in terms of employment of both factors. This results from the fact that when intermediate inputs are gross complements in the production of final output, $\varepsilon < 1$, the laggard sector will determine the asymptotic growth rate of final output.

We define aggregate capital per worker deflated by the TFP level of the technologically laggard sector,

[21] Notice that if $m_2 = m_1 = m$, then by normalization we can write $M_1 = M_2 = M$, and using Propositions 1 and 2, we can treat κ and λ as functions of χ. We would then face a single first order differential equation. In the derivations that follow we deal with the more general case where $m_2 > m_1 \geq 0$. With sector-specific rates of TFP growth, we face a non-stationary system. It turns out, however, that we can analyze convergence to a constant growth path by studying a system of two differential equations that are stationary, see below.

$$\chi(t) \equiv \frac{K(t)}{L(t)\, M_1(t)} \tag{15}$$

Then, using (3) we reach,

$$\widehat{\chi} \equiv \frac{\dot{\chi}(t)}{\chi(t)} = \frac{\dot{K}(t)}{K(t)} - n - m_1 = \frac{I(t)}{K(t)} - n - m_1 - \delta = s\frac{Y(t)}{K(t)} - n - m_1 - \delta \tag{16}$$

While the dynamic equation (16) is common to all three models, we will see below that the expression for $Y(t)/K(t)$ differs across specifications.

4.1 Special Case 1: Dynamics of the Acemoglu-Guerrieri model

Recall that this case arises when $\sigma_1 = \sigma_2 = 1$ and $\alpha_2 > \alpha_1$. Given initial conditions for the state variables, $\chi(0)$ and $A(0) \equiv M_2(0)/M_1(0)$, the initial sectoral allocation of capital, $\kappa(0) = \mathcal{K}(\chi(0), A(0))$, is uniquely determined by versions of the (CC) and (LM) curves with sectoral Cobb-Douglas technologies. The following proposition summarizes the dynamic behavior of the AG model.

Proposition 5 *Under the stated assumptions and given the initial conditions,* $\chi(0) = \chi_0$ *and* $A(0) = A_0$ *and consequently* $\kappa(0) = \mathcal{K}(\chi_0, A_0)$*, the solution to the dynamic problem satisfies the following system of differential equations.*

$$\hat{\chi}(t) = s\eta(t)\ \lambda(t)^{1-\alpha_1}\kappa(t)^{\alpha_1}\chi(t)^{\alpha_1-1} - (\delta + n + m_1) \tag{17}$$

$$\hat{\kappa}(t) = \frac{(1-\kappa(t))(\alpha_2-\alpha_1)(\hat{\chi}(t)+m_2)}{(1-\varepsilon)^{-1} + (\alpha_2-\alpha_1)(\lambda(t)-\kappa(t))} \tag{18}$$

where $\eta(t) = \gamma^{\frac{\varepsilon}{\varepsilon-1}}\left[1 + \frac{\alpha_1}{\alpha_2}\frac{(1-\kappa(t))}{\kappa(t)}\right]^{\frac{\varepsilon}{\varepsilon-1}}$ *and* $\lambda(t) = \left(\frac{1-\alpha_2}{1-\alpha_1}\frac{\alpha_1}{\alpha_2}\frac{(1-\kappa(t))}{\kappa(t)} + 1\right)^{-1}$.

Proof. See Appendix A. ■

Let us define the following growth rates for the variables of interest (using the superscript ss for their asymptotic, steady state, counterparts),

$$\frac{\dot{L}_s(t)}{L_s(t)} \equiv n_s(t), \quad \frac{\dot{K}_s(t)}{K_s(t)} \equiv z_s(t), \quad \frac{\dot{Y}_s(t)}{Y_s(t)} \equiv g_s(t), \quad \text{for } s = 1, 2,$$

$$\frac{\dot{K}(t)}{K(t)} \equiv z(t), \quad \frac{\dot{Y}(t)}{Y(t)} \equiv g(t)$$

We define a "constant growth path" (CGP) as a solution along which the aggregate capital-output ratio, $\frac{K}{Y}$, is constant.[22] Then we have the following characterization of the unique (non-trivial) CGP.

Theorem 1 *Under the stated assumptions, there exists a unique (non-trivial) CGP with the following properties,*

$$\begin{aligned}
\chi^{ss} &= \left(\frac{s\gamma^{\frac{\varepsilon}{\varepsilon-1}}}{\delta+n+m_1}\right)^{\frac{1}{1-\alpha_1}}, \qquad \kappa^{ss}=\lambda^{ss}=1, \\
g^{ss} &= z^{ss}=g_1^{ss}=z_1^{ss}=n+m_1, \\
g_2^{ss} &= g^{ss}+\varepsilon\left(1-\alpha_2\right)\left(m_2-m_1\right), \qquad z_2^{ss}=g^{ss}-\left(1-\varepsilon\right)\left(1-\alpha_2\right)\left(m_2-m_1\right), \\
n_1^{ss} &= n, \qquad n_2^{ss}=n-\left(1-\varepsilon\right)\left(1-\alpha_2\right)\left(m_2-m_1\right).
\end{aligned} \tag{19}$$

The steady state associated with this CGP is locally stable.[23]

Proof. See Appendix A. ■

A couple of results require further comment. First, the two intermediate sectors grow at different rates even along the CGP. In this sense growth is non-balanced. This unbalanced growth arises from the differential TFP growth rates across sectors. Notice that if both sectoral TFP grew at the same rate, $m_1=m_2=m$, then the steady state sectoral per capita growth rates would coincide with the exogenous rate of TFP growth, m.[24] Second, although the fractions of inputs in the asymptotically dominant sector 1, κ and λ, tend in the limit to one, sector 2 grows faster than the rest of the economy along the CGP.[25] In this sense both sectors will be permanently operative although the shares of capital and employment in sector 2 become asymptotically trivial. Finally, as the economy asymptotically reaches the CGP the process of sectoral reallocation ceases.

[22] Our definition is equivalent to that of AG and Kongsamut et al. (2001). The former paper requires a constant consumption growth rate in a model of endogenous saving. The latter defines a "generalized balanced growth path" as a trajectory along which the real interest rate is constant.

[23] One can talk about the "stability" of this system in the following sense: Suppose that, given $(\chi(0), A(0))$, and hence $\kappa(0)=\mathcal{K}(\chi(0),A(0))$, we can show that $\chi(t)\to\chi^{ss}$ and $\kappa(t)\to 1$. Perturb $A(0)$ (e.g., consider a sudden jump in A at time zero, from A_0 to $A_0^{\#}$). Then $\kappa(0)=\mathcal{K}(\chi(0),A(0))$ jumps to a new value, $\mathcal{K}(\chi(0),A_0^{\#})$. With this new value of $\kappa(0)$, do $(\chi(t),\kappa(t))$ still converge to $(\chi^{ss},1)$? Theorem 1 shows that the system is stable in the sense that the answer to the preceding question is in the affirmative.

[24] Notice that the condition for balanced growth in the original AG model is $m_1/\alpha_1=m_2/\alpha_2$ rather than $m_1=m_2$ since AG specify TFP as Hicks neutral while in our equation (4) we model TFP as Harrod neutral.

[25] Takahashi (2014) explores a version of the AG model where capital goods are produced only in one of the two sectors. In contrast to AG, this economy converges to a stable growth path, on which the growth rate of aggregate output is determined by that of the sector that exhibits faster technical change.

4.2 Special Case 2: Dynamics of the simplified Ngai-Pissarides model

Recall that this case arises when $\sigma_1 = \sigma_2 = 1$ and $\alpha_2 = \alpha_1 = \alpha$. As before we restrict attention to the case where $\varepsilon < 1$. Again, let $m_2 > m_1$ so that sector 1 is the asymptotically dominant sector for the same reasons as in the previous model.

Notice that in this case, given the initial levels of sectoral total factor productivity, $A(0) \equiv M_2(0)/M_1(0)$, the initial sectoral allocation of capital, $\kappa(0) = \mathcal{K}(A(0))$, is uniquely determined by the (LM) curve given that the (CC) curve implies $\lambda = \kappa$. The following proposition summarizes the dynamic behavior of the simplified NP model.

Proposition 6 *Under the stated assumptions and given the initial conditions, $\chi(0) = \chi_0$ and $A(0) = A_0$ and consequently $\kappa(0) = \mathcal{K}(A_0)$, the competitive equilibrium satisfies the following pair of differential equations,*

$$\hat{\chi}(t) = s\xi(t)\chi(t)^{\alpha_1 - 1} - (\delta + n + m_1) \tag{20}$$

$$\hat{\kappa}(t) = (1 - \kappa(t))(1 - \alpha)(1 - \varepsilon)(m_2 - m_1) \tag{21}$$

where $\xi(t) \equiv \left[\gamma\kappa(t)^{\frac{\varepsilon-1}{\varepsilon}} + (1-\gamma)\left(\left(\frac{M_2(t)}{M_1(t)}\right)^{1-\alpha}(1-\kappa(t))\right)^{\frac{\varepsilon-1}{\varepsilon}}\right]^{\frac{\varepsilon}{\varepsilon-1}}.$

Proof. See Appendix A. ■

Let's turn now to the characterization of the constant growth path.

Theorem 2 *Under the stated assumptions, there exists a unique (non-trivial) CGP with the following properties,*

$$\begin{aligned}
\chi^{ss} &= \left(\frac{s\gamma^{\frac{\varepsilon}{\varepsilon-1}}}{\delta + n + m_1}\right)^{\frac{1}{1-\alpha_1}}, \qquad \kappa^{ss} = \lambda^{ss} = 1, \\
g^{ss} &= z^{ss} = g_1^{ss} = z_1^{ss} = n + m_1, \\
g_2^{ss} &= g^{ss} + \varepsilon(1-\alpha)(m_2 - m_1), \qquad z_2^{ss} = g^{ss} - (1-\varepsilon)(1-\alpha)(m_2 - m_1), \\
n_1^{ss} &= n, \qquad n_2^{ss} = n - (1-\varepsilon)(1-\alpha)(m_2 - m_1).
\end{aligned} \tag{22}$$

The steady state associated with this CGP is locally stable (in the sense explained in footnote 23).

Proof. Notice that equation (21) does not contain the variable χ and thus can be analyzed independently. This equation gives $\dot{\kappa} = \kappa\,(1-\kappa)\,(1-\alpha)\,(1-\varepsilon)\,(m_2 - m_1)$ which is the familiar logistic equation with two steady states, $\kappa_1^{ss} = 1$ and $\kappa_2^{ss} = 0$. The first one is asymptotically stable, and the second one is asymptotically unstable. As $\kappa \to 1, \xi \to \gamma^{\frac{\varepsilon}{\varepsilon-1}}$ and $\chi \to \chi^{ss}$. The CGP associated with the steady state $(1, \chi^{ss})$ is locally stable since, evaluated at this point, $\dfrac{\partial\widehat{\kappa}}{\partial\kappa} = (\alpha-1)\,(1-\varepsilon)\,(m_2-m_1) < 0$ and $\dfrac{\partial\widehat{\chi}}{\partial\chi} = (\alpha-1)\,(\delta+n+m_1) < 0$. ■

As in the AG model, growth is non-balanced with the two sectors growing permanently at different rates, the fractions of capital and labor allocated to the sector with high TFP growth asymptotically vanish, and structural change asymptotically ceases as the economy approaches the CGP. These last two features are specific to our simplified version of the NP model. In NP's original framework, with several sectors producing consumption goods and one sector producing consumption and capital goods, structural change among the consumption-producing sectors still takes place along the CGP and the fractions of resources allocated to the capital producing sector never vanish.

4.3 Dynamics of the model of unequal sectoral capital-labor substitution

Recall that this case arises when $\sigma_2 \neq \sigma_1$. Furthermore, for tractability, we impose the restriction that $\sigma_1 = 1$, and allow σ_2 to be either larger or smaller than one. To analyze the effects of differences in the sectoral elasticity of substitution in isolation, we assume that $\varepsilon = 1$ and that sectors are symmetric in all other respects, i.e. that $\alpha_2 = \alpha_1 = \alpha$ and $M_1(t) = M_2(t) = M(t)$, which implies $m_1 = m_2 = m$. Combining the (CC) curve, (55) and (15) we reach the following one-to-one relationship between χ and κ,

$$\chi(\kappa) = (\gamma(1-\alpha))^{\frac{\sigma_2}{\sigma_2-1}} \frac{(1-\kappa)^{\frac{1}{\sigma_2-1}}}{(\kappa-\alpha\gamma)\,(\kappa\,(1-\gamma(1-\alpha))-\alpha\gamma)^{\frac{1}{\sigma_2-1}}} \tag{23}$$

where $\chi'(\kappa) < 0$ (resp. $\chi'(\kappa) > 0$) for all $\kappa \in (\underline{\kappa}, 1)$ if $\sigma_2 > 1$ (resp. $\sigma_2 < 1$). Furthermore, it is worth noticing that when $\sigma_2 > 1$, $\chi(1) = 0$ and $\lim_{\kappa\to\underline{\kappa}} \chi(\kappa) = \infty$, and when $\sigma_2 < 1$, $\chi(\underline{\kappa}) = 0$ and $\lim_{\kappa\to 1} \chi(\kappa) = \infty$.

The following proposition summarizes the dynamic behavior of this model.

Proposition 7 *Under the stated assumptions, given the initial condition $\chi(0) = \chi_0$, the competitive equilibrium path satisfies the following differential equation:*

$$\dot{\kappa} = \frac{sD\pi(\kappa) - (\delta+m+n)}{H(\kappa)} \tag{24}$$

where

$$D \equiv \left[\frac{1-\gamma}{\gamma}\right]^{\frac{(1-\gamma)\sigma_2}{\sigma_2-1}} \left(\gamma\left(1-\alpha\right)\right)^{-\frac{(1-\alpha)\gamma}{\sigma_2-1}}, \tag{25}$$

$$\pi(\kappa) \equiv \frac{\kappa^{(\sigma_2-\gamma)/(\sigma_2-1)}\left(\kappa\left(1-\gamma(1-\alpha)\right)-\alpha\gamma\right)^{\frac{\gamma(1-\alpha)}{\sigma_2-1}}}{(1-\kappa)^{\frac{1-\alpha\gamma}{\sigma_2-1}}}, \tag{26}$$

and

$$H(\kappa) \equiv -\left(\frac{1}{\sigma_2-1}\right)\left[\frac{(1-\gamma+\alpha\gamma)(\kappa-\alpha\gamma)+\sigma_2\left(\kappa(1-\gamma(1-\alpha))-\alpha\gamma\right)}{\left(\kappa-\alpha\gamma\right)\left(\kappa(1-\gamma(1-\alpha))-\alpha\gamma\right)}\right] \tag{27}$$

where $H(\kappa) < 0$ *(resp.* $H(\kappa) > 0$*) for all* $\kappa \in (\underline{\kappa}, 1)$ *if* $\sigma_2 > 1$ *(resp.* $\sigma_2 < 1$*).*

Proof. See Appendix A. ■

Let's turn now to the characterization of the constant growth path.

Theorem 3 *Under the stated assumptions, there exists a unique (non-trivial) CGP that satisfies*

$$\begin{aligned}
\pi(\kappa^{ss}) &= \frac{\delta+m+n}{sD}, \qquad \lambda^{ss} = \frac{\gamma\left(1-\alpha\right)\kappa^{ss}}{\kappa^{ss}-\alpha\gamma}, \\
\chi^{ss} &= (\gamma(1-\alpha))^{\frac{\sigma_2}{\sigma_2-1}} \frac{(1-\kappa^{ss})^{\frac{1}{\sigma_2-1}}}{\left(\kappa^{ss}-\alpha\gamma\right)\left(\kappa^{ss}\left(1-\gamma(1-\alpha)\right)-\alpha\gamma\right)^{\frac{1}{\sigma_2-1}}} \\
g^{ss} &= z^{ss} = g_1^{ss} = g_2^{ss} = z_1^{ss} = z_2^{ss} = n+m \qquad n = n_1^{ss} = n_2^{ss}.
\end{aligned}$$

The steady state associated with this CGP is locally stable.

Proof. Notice that (24) is an autonomous differential equation with a unique (non-trivial) steady state, κ^{ss}. The CGP associated with this steady state is locally stable since, evaluated at that point, $\frac{\partial\dot{\kappa}}{\partial\kappa} = \frac{sD\pi'(\kappa^{ss})}{H(\kappa^{ss})} < 0$. ■

This theorem has several interesting implications. First, since the sectoral TFP growth rates are identical, both sectors grow at the same rate along the CGP which, of course, is the same as the growth rate of the aggregate economy. Second, the steady state fractions of employment and capital are strictly positive in both sectors. As opposed to the previous models where the fractions of employment and capital in the sector that sheds resources asymptotically vanish, in this model both sectors reach the CGP with non-trivial shares of employment and capital.[26] Third, it is worth noticing that along the CGP the capital-output ratio and the rental rate are constant, and as a result, so is the share of capital in national income, while the wage rate grows at the exogenous rate of TFP growth, m. Finally, as in the AG model, once the economy reaches the CGP the process of sectoral reallocation comes to an end. This is the case since along such a path capital per unit of effective labor, χ, is constant and therefore the incentives for sectoral reallocations induced by capital deepening and technological change perfectly cancel out, since they are exactly equal but work in opposite directions. This becomes clear once one notices that (13a)-(13e) imply that $\frac{\partial \lambda}{\partial k}\frac{k}{\lambda} = -\frac{\partial \lambda}{\partial M}\frac{M}{\lambda}$ and $\frac{\partial \kappa}{\partial k}\frac{k}{\kappa} = -\frac{\partial \kappa}{\partial M}\frac{M}{\kappa}$.[27]

During the transition to a balanced growth path, differences in capital-labor substitutability across sectors can thus lead to sectoral reallocations. For instance, during a transition "from below", along which capital, K, grows faster than effective labor, ML, the more flexible sector will substitute towards capital, the input that becomes relatively abundant. Hence, the more flexible sector will become more capital intensive, and the less flexible sector more labor intensive. As a consequence, the capital-labor ratio in the more flexible sector will grow faster than its counterpart in the less flexible sector.

5 Optimal growth under unequal sectoral capital-labor substitution

In this section we extend the previous analysis to the case of optimal saving. Let C and I denote consumption and gross investment respectively. Then

$$Y = C + I \tag{28}$$

$$\dot{K} = Y - C - \delta K = I - \delta K \tag{29}$$

Let c be consumption per unit of effective labour,

$$c \equiv \frac{C}{ML} = \frac{C}{K}\left(\frac{K}{ML}\right) = \frac{C}{K}\chi \tag{30}$$

[26]Notice that when $m_1 = m_2 = 0$ the AG model also reaches non-trivial steady-state allocations of capital and labor for both sectors. Nonetheless in this case growth eventually ceases. Similarly, non-trivial allocations along the CGP of the simplified NP model occur when $m_1 = m_2 = m$, but in this case there is no structural change and the initial fractions of capital and labor allocated to each sector remain constant forever.

[27]The same conclusion could be reached in terms of the ratio of the wage per unit of effective labor to the rental rate that drives sectoral reallocations. Once χ is constant, this ratio is also constant, so structural change stops.

Then using (64) from Appendix A we reach,

$$\frac{I}{K} = \frac{Y}{K} - \frac{C}{K} = D\pi(\kappa) - \frac{c}{\chi} \tag{31}$$

and therefore the law of motion of capital per unit of effective labour becomes,

$$\dot{\chi} = D\pi(\kappa)\chi - c - (\delta + m + n)\,\chi \tag{32}$$

Let the instantaneous utility function, $U\left(C/L\right)$, take the familiar CRRA specification where $1/\left(1-\mu\right) > 0$ is the intertemporal elasticity of substitution of consumption. Then the discounted life-time welfare of the representative household is

$$\int_0^\infty e^{-\beta t} LU\left(\frac{C}{L}\right) dt = \int_0^\infty e^{-\beta t}\left(\frac{c^\mu}{\mu}\right) LM^\mu dt = \int_0^\infty e^{-(\beta-n-\mu m)t}\left(\frac{c^\mu}{\mu}\right) dt \tag{33}$$

where β is the rate of time preference and we have used the fact that population and TFP grow at the exogenous rates n and m respectively.

The solution to the optimal growth problem amounts to finding the time path for c that maximizes (33) subject to (32). Notice that from (23), κ is a function of χ and we can define

$$f\left(\chi\right) \equiv D\pi(\kappa(\chi))\chi \tag{34}$$

Therefore (32) becomes

$$\dot{\chi} = f\left(\chi\right) - c - (\delta + m + n)\,\chi \tag{35}$$

Thus, this optimization problem reduces to the standard optimal growth problem if $f\left(\chi\right)$ is a strictly concave and increasing function of χ with $f\left(0\right) = 0$. Appendix B shows that $f\left(\chi\right)$ does indeed satisfy these properties.

Define

$$\rho \equiv \beta - n - \mu m$$

and assume that ρ is positive. Let ν be the shadow price of χ. The Hamiltonian for this problem is

$$H = \frac{c^\mu}{\mu} + \nu\left[f\left(\chi\right) - c - (\delta + m + n)\chi\right]$$

The necessary conditions are

$$c^{\mu-1} = \nu \tag{36}$$

$$\dot{\nu} = \nu\left[\rho + \delta + m + n - f'(\chi)\right] \tag{37}$$

together with the transversality condition, $\lim_{t\to\infty} \chi\nu \exp\left(-\rho t\right) = 0$.

Combining (36) and (37) we reach the familiar consumption Euler equation,

$$\dot{c} = \frac{c}{1-\mu} \left[f'(\chi) - \rho - \delta - m - n \right] \tag{38}$$

that together with (35), the initial condition, $\chi(0) = \chi_0$, and the transversality condition fully describe the dynamic evolution of the economy. The steady state satisfies,

$$f'(\chi^{ss}) = \rho + \delta + m + n \tag{39}$$

$$c^{ss} = f(\chi^{ss}) - (\delta + m + n) \chi^{ss} \tag{40}$$

and the standard analysis applies.

6 Suggestive evidence

Although the focus of the paper is theoretical, we use this section to provide some illustrative evidence consistent with the core mechanism in the model of unequal sectoral capital-labor substitution. We also show how the aggregate elasticity of substitution depends on model parameters and the economy's structural composition.

6.1 Structural change out of agriculture and capital intensity

In the introduction, we mentioned that in the U.S., structural change out of agriculture was accompanied by important differences in the growth rates of sectoral capital-labor ratios: from the end of World War II to recent years, the capital-labor ratio in agriculture grew on average by 3.2% per year, compared to 1.7% in non-agriculture. At the same time, the share of value added produced in agriculture declined from 10% to around 1% of GDP.

Clearly, structural change has not only taken place in the U.S.. In fact, the process of economic development is always and everywhere characterized by substantial reallocations of resources out of agriculture. As a result of this process, the differences in sectoral structure between developed and developing countries are staggering. On the one hand, rich countries, such as the U.S., the U.K. or Belgium employ less than 3% of their labor force in agriculture, while on the other hand, poor countries such as Nepal, Burundi or Niger have employment shares in agriculture in excess of 90%.

These differences in employment shares are compounded by large differences in labor productivity and capital intensity. As Restuccia, Yang, and Zhu (2008), Chanda and Dalgaard (2008), and Gollin, Lagakos, and Waugh (forthcoming) report, the differences in agricultural labor productivity between rich and poor countries are twice as large as those in aggregate labor productivity. Mundlak (2000) finds that the cross-country distributions of various measures of investment and capital show much larger dispersion in agriculture compared to the rest of the economy.

Table 1. Key variables across countries.

	Y / L	K^{na} / L^{na}	K^{a} / L^{a}	L^{a} / L	K^{a} / K
Rich 5	18,000	82,273	100,318	5.0%	5.7%
Poor 5	807	6,482	191	78.2%	10.8%
Ratio	22	13	525	1/17	1/2
Mean	8,239	36,954	26,790	28.3%	7.4%
Std. Dev.	6,068	29,346	33,992	23.6%	3.7%
Min	529	1,297	23	2.0%	1.8%
Max	20,000	99,492	125,621	84.0%	16.4%
Coeff. Var	0.74	0.19	1.27	0.84	0.5

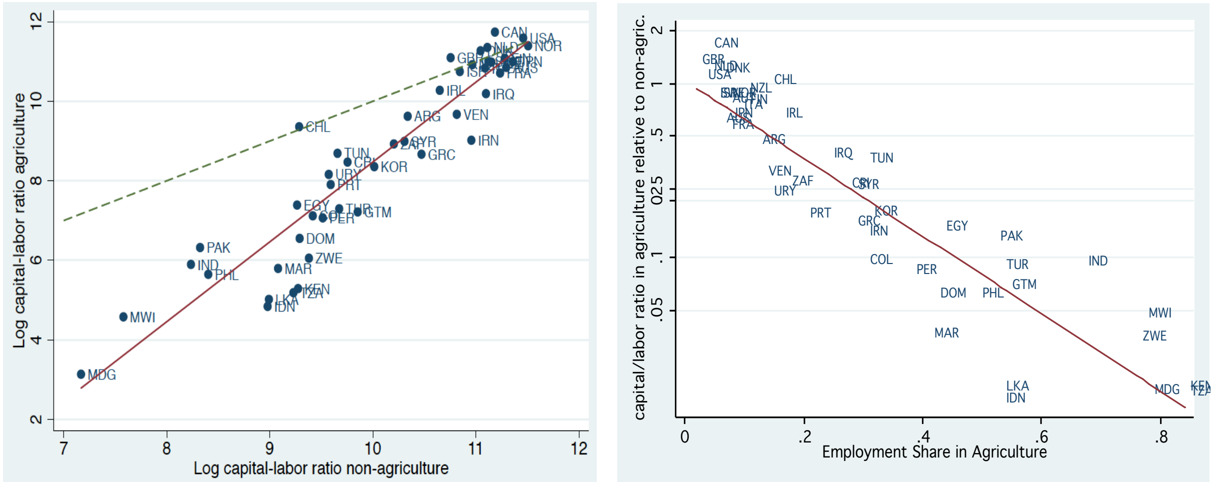
Sources: Crego et al. (2000), Duarte and Restuccia (2010) and GGDC Total economy database

This is clear in Table 1, which combines data on sectoral capital stocks for 50 countries collected by Crego et al. (2000) with data on labor allocations from Duarte and Restuccia (2010) to illustrate the cross-country variation in sectoral capital-labor ratios. The table shows that in rich countries, agriculture is on average somewhat more capital-intensive than the rest of the economy, while in the poorest economies, agriculture uses very little capital at all.[28] The size of these differences in capital per worker across sectors and countries is stunning. In the 5 richest countries in our sample, capital per worker outside of agriculture is 13 times larger than in the 5 poorest countries. At the same time, it is 500 times larger in the agricultural sector.[29] Figure 6(a) plots capital-labor ratios in agriculture versus those in non-agriculture. Since the correlation coefficient between non-agricultural capital-labor ratios and income per capita is above 0.95 one can think of this variable as a measure of development. A simple regression of agricultural capital per worker on its non-agricultural counterpart suggests that a one percentage point increase in non-agricultural capital-labor ratio is associated with a two percentage point increase in agricultural capital per worker.

[28]The five richest countries in our sample are the U.S., Canada, Denmark, Norway and Sweden, while the five poorest ones are Tanzania, Malawi, Madagascar, Kenya and India. Data is for 1990.

[29]An example may help visualizing how these differences come about. For instance, the Lexion 590R, the world's largest combine harvester, has the capacity to harvest 1,800 bushels of wheat per hour. This capacity is equivalent to 540 man-hours.

Figure 5: Sectoral capital labor ratios, structural change, and development



(a) Sectoral capital-labor ratios: agriculture vs. non-agriculture

(b) Relative capital-labor ratios vs. agricultural employment share

Notes: Data sources: Crego et al. (2000) and Duarte and Restuccia (2010). In both panels, the solid line is an OLS regression line. The dashed line in panel (a) is the 45-degree line.

Figure 6(b) relates the same pattern more closely to structural change. It plots the log of the capital-labor ratio in agriculture relative to that in non-agriculture against the fraction of persons engaged in agriculture. The relationship is very clearly negative. A regression of the log relative capital-labor ratio on the share of employment in agriculture shows that a decline in the share of employment by one percentage point goes along with an increase in the relative capital-labor ratio by 5 percent. This implies that the relative capital-labor ratio doubles every time the employment share in agriculture declines by 14 percentage points.

Given the important cross-country variation in wage to rental rate ratios, one may interpret the large sectoral variation in capital-labor ratios as the result of a relatively high elasticity of substitution between inputs in agriculture. This interpretation is consistent with the evidence provided by Herrendorf et al. (2014). These authors estimate sectoral CES production functions using postwar U.S. data. They report an estimate of the elasticity of substitution between capital and labor in agriculture of 1.58, twice as high as their estimates for manufacturing and services.[30] Rosenzweig (1988) implicitly acknowledges this

[30] In 1950, the value-added share of agriculture in the U.S. was already barely 10%. Therefore, most of the sectoral reallocations in Herrendorf et al.'s period of analysis took place between manufacturing and services. Since their estimates suggest that the elasticities of substitution in those two sectors are very similar (0.75 for services and 0.8 for manufacturing), it is not surprising that they conclude that differences in sectoral elasticities of substitution are of secondary importance for the postwar U.S. structural transformation. Nonetheless, these differences may well be important for economies that still employ a large fraction of resources in the agricultural sector.

substitution capability of agricultural production when arguing that obstacles to migration out of agriculture depress rural wages, inducing farmers to substitute cheap labor for capital and intermediate inputs. Along similar lines, Manuelli and Seshadri (2014) provide evidence on the impact of low labor costs on the slow rate of adoption of tractors in U.S. agriculture between 1910 and 1940. As long as agricultural wages were low, producers found it more profitable to operate the labor-intensive horse technology rather than the capital-intensive tractor.

All this evidence suggests that the degree of flexibility in agricultural production may be important for understanding the observed cross-country variation in sectoral capital-labor ratios and in sectoral labor productivities. In this view, agricultural production is labor-intensive in poor countries, which have relatively low wage to rental rate ratios. As countries develop and capital per worker increases, the flexible agricultural sector reacts strongly to the increase in the wage to rental rate ratio, substituting away from now relatively more expensive labor and into now relatively cheaper capital. As a result, agricultural capital-labor ratios increase faster with development than non-agricultural ones. This view is consistent both with the U.S. evidence discussed in the introduction and with the patterns illustrated in Figure 3.

6.2 The aggregate elasticity of substitution

Evaluating the response of the economy-wide capital-labor ratio to changes in the factor-price ratio requires a measure of the degree of substitutability between capital and labor at the aggregate level. In recent years, several studies have estimated this aggregate elasticity using U.S. time series data (see e.g. Antras (2004), Herrendorf et al. (2014), Leon-Ledesma et al. (2015)), cross-sectional U.S. data (Oberfield and Raval 2014) or cross-country data (Karabarbounis and Neiman 2014). Nonetheless, in our two-sector framework, this elasticity is not a “deep” parameter, but depends on other parameters and on allocations. In what follows, we borrow from the dual approach developed in Jones (1965) to characterize this aggregate elasticity.

Lemma 2 The aggregate elasticity of substitution. *The aggregate elasticity of substitution, σ, is a weighted average of the three primary elasticities: the elasticity of substitution between the two intermediate inputs in the production of the final good, ε, and the two sectoral elasticities, σ_1 and σ_2:*

$$\sigma = \gamma_0 \varepsilon + \gamma_1 \sigma_1 + \gamma_2 \sigma_2 \tag{41}$$

where the weights,

$$\gamma_0 \equiv (\epsilon_2 - \epsilon_1)(\lambda - \kappa)$$
$$\gamma_1 \equiv \lambda\epsilon_1 + \kappa(1 - \epsilon_1)$$
$$\gamma_2 \equiv (1 - \lambda)\epsilon_2 + (1 - \kappa)(1 - \epsilon_2)$$

add up to one and, as before, ϵ_s is the capital income share in sector s.

Proof: See Appendix A. ■

Although the primary elasticities are constant, in general the aggregate elasticity of substitution varies with the sectoral composition of output. As an exception, in the simplified NP model, where $\sigma_1 = \sigma_2 = 1$ and $\epsilon_1 = \epsilon_2 = \alpha$, this aggregate elasticity is constant. Intuitively, if factor income shares in both sectors are equal, then $\gamma_0 = 0$, so that the aggregate elasticity of substitution is independent of the elasticity of substitution of the final sector. As a result, the aggregate elasticity reduces to a weighted average of the sectoral elasticities. It is constant when both sectoral elasticities coincide. The aggregate elasticity of substitution in the AG model, where $\sigma_1 = \sigma_2 = 1$, is given by

$$\sigma = 1 + (\alpha_2 - \alpha_1)(\lambda - \kappa)(\varepsilon - 1) < 1 \Longleftrightarrow \varepsilon < 1. \tag{42}$$

In this case the aggregate elasticity, which changes with the process of structural change, is below one if the elasticity of substitution between inputs in the final sector is below one. Notice that this is so since (41) reduces to a weighted average of the elasticity in the final sector, ε, and 1.

Finally, under the parameterization used in the dynamic analysis of the model of unequal sectoral capital-labor substitution, where $\sigma_2 \neq \sigma_1 = \varepsilon = 1$, the aggregate elasticity of substitution is given by

$$\sigma = \gamma_0 + \gamma_1 + \gamma_2\sigma_2 < 1 \Longleftrightarrow \sigma_2 < 1, \tag{43}$$

so that σ is a weighted average of 1 and one of the sectoral elasticities, σ_2. It is thus σ_2 which determines whether the aggregate elasticity lies above or below one. Since γ_2 varies with development, here, too, the aggregate elasticity of substitution is not constant.

This feature of the model is in line with evidence from Duffy and Papageorgiou (2000), who use World Bank data on capital stocks for 82 countries over 28 years to estimate the parameters of an aggregate CES production function. They do so using four subsamples and find that the elasticity of substitution increases with development. In the view of our model, differences in sectoral elasticities coupled with the secular process of capital accumulation might lie behind this relationship between flexibility in production and the level of development.

7 Conclusions

We have developed a two-sector model where differences in the sectoral elasticity of substitution between capital and labor lead to a process of structural change. The mechanism behind this model is simple. As the wage to rental rate ratio changes, the more flexible sector – the sector with a higher elasticity of substitution between capital and labor – is in a better position to take advantage of these changes than the less flexible one. As a result, if the final-goods sector is Cobb-Douglas, the flexible sector increases its share in the aggregate capital stock and reduces its share in employment as capital accumulates. Despite this process of structural change, the economy eventually reaches a constant growth path where the fractions of employment and capital in both sectors are positive and constant. The sectoral reallocations driven by unequal capital-labor substitution contrast with those driven by differences in sectoral rates of total factor productivity growth as in the simplified NP model or by differences in capital intensity as in the AG model. In these models, a sector either sheds or absorbs both capital and labor. Additionally, we show how our main results extend to a Ramsey environment, where saving is endogenous, and provide some suggestive evidence on the relevance of the mechanism stressed by this unequal capital-labor substitution model.

The main contribution of this paper is theoretical and qualitative. In terms of future, more applied work, our model provides a promising channel for explaining structural change in situations where sectoral capital-labor ratios grow at different rates or situations where sectoral labor income shares evolve differently, both pointing to differences in the elasticity of substitution across sectors. Another natural extension would be to explore the quantitative importance of the mechanism stressed by the model for actual patterns of structural change. This requires a more realistic framework that incorporates other sources of sectoral reallocation that seem to be important, such as non-homothetic preferences and differential TFP growth. Furthermore, it would be interesting to explore numerically whether structural change is consistent with quasi-balanced growth, i.e. whether changes in factor allocations are first-order in magnitude when changes in the great ratios are already of second-order. Some of these issues are treated in our companion paper (Alvarez-Cuadrado, Long and Poschke 2014), which focusses on changes in labor income shares in the U.S. from 1960 to 2005, a period of intense structural change from manufacturing to services.

Finally, although there have been several attempts to estimate the aggregate elasticity of substitution, our analysis suggests that, in general, this aggregate elasticity is a time-varying combination of deeper structural parameters, particularly of the sectoral elasticities of substitution. In this sense, another natural extension of this project would be to pursue the estimation of these elasticities at different levels of aggregation, along the lines of Herrendorf et al. (2014). Through the link between sectoral and aggregate elasticities of substitution given by equation (41), this exercise would also allow recovering an estimate of this aggregate elasticity. Furthermore, these sectoral elasticities could also be used to test some of the predictions of the model. In principle, existing datasets, such as the 35-industry KLEM developed by Dale W. Jorgenson and the EU-KLEMS gathered by the Groningen Growth and Development Center, provide the sectoral level data required for these estimations.

Appendix A

Proof of Lemma 1. Firstly, $\epsilon_{CC,\lambda} < 0$ and $\epsilon_{CC,k} > 0$. Hence, to compensate the increase in the left hand side of (CC) brought about by an increase in k, λ needs to increase, for any κ. Secondly, $\epsilon_{CC,\lambda} < 0$ and $\epsilon_{CC,M1} + \epsilon_{CC,M2} = \frac{1}{\sigma_2} - \frac{1}{\sigma_1} < 0$, as $\sigma_2 > \sigma_1$. Hence, to compensate the decline in the left hand side of (CC) brought about by a proportional increase in M_1 and M_1, λ needs to decline, for any κ. ■

Proof of Proposition 1.

Determination of the condition distinguishing a. and b. (LM) increases in k if

$$\epsilon_{LM,k} = \left(\frac{1}{\sigma_1} - \frac{1}{\varepsilon}\right)\epsilon_1 - \left(\frac{1}{\sigma_2} - \frac{1}{\varepsilon}\right)\epsilon_2 > 0. \tag{P1.A}$$

Determination of the condition distinguishing a.i. and a.ii. Consider a situation where k changes by a proportion $g(k)$, and κ remains unchanged. Then (CC) requires a change in λ of

$$g(\lambda_{CC}) = \frac{\frac{1}{\sigma_1} - \frac{1}{\sigma_2}}{\frac{1}{\sigma_1} + \frac{1}{\sigma_2}\frac{\lambda}{1-\lambda}} g(k).$$

For (LM) to hold, a change in λ of

$$g(\lambda_{LM}) = \frac{\left(\frac{1}{\sigma_1} - \frac{1}{\varepsilon}\right)\epsilon_1 - \left(\frac{1}{\sigma_2} - \frac{1}{\varepsilon}\right)\epsilon_2}{\frac{1}{\sigma_1}\epsilon_1 + \frac{1}{\sigma_2}\epsilon_2\frac{\lambda}{1-\lambda} + \frac{1}{\varepsilon}\left((1-\epsilon_1) + (1-\epsilon_2)\frac{\lambda}{1-\lambda}\right)} g(k)$$

is needed. Under the assumption that $\sigma_2/\sigma_1 > \epsilon_2/[(1 - \frac{\sigma_1}{\varepsilon})\epsilon_1 + \frac{\sigma}{\varepsilon}\epsilon_2]$, this is positive. Comparing these two expressions implies that $g(\lambda_{CC}) > g(\lambda_{LM})$ if

$$\left(\frac{1}{\sigma_1} - \frac{1}{\sigma_2}\right)\left(\frac{1}{\sigma_1}\epsilon_1 + \frac{1}{\sigma_2}\epsilon_2\frac{\lambda}{1-\lambda} + \frac{1}{\varepsilon}\left((1-\epsilon_1) + (1-\epsilon_2)\frac{\lambda}{1-\lambda}\right)\right)$$
$$> \left(\frac{1}{\sigma_1} + \frac{1}{\sigma_2}\frac{\lambda}{1-\lambda}\right)\left(\left(\frac{1}{\sigma_1} - \frac{1}{\varepsilon}\right)\epsilon_1 - \left(\frac{1}{\sigma_2} - \frac{1}{\varepsilon}\right)\epsilon_2\right).$$

After some tedious algebra, this condition becomes

$$\frac{\epsilon_2 - \epsilon_1}{\sigma_1\sigma_2} - \frac{1-\epsilon_1}{\varepsilon\sigma_2} + \frac{1-\epsilon_2}{\varepsilon\sigma_1} > 0$$

or

$$\epsilon_1 - \epsilon_2 < \frac{(1-\epsilon_2)\,\sigma_2 - (1-\epsilon_1)\,\sigma_1}{\varepsilon}.$$

Defining $\vartheta \equiv \epsilon_1 - \epsilon_2$, substituting out ϵ_2 using ϑ and ϵ_1, and solving for ϑ then yields

$$\vartheta\left(\varepsilon - \sigma_2\right) < \left(1 - \epsilon_1\right)\left(\sigma_2 - \sigma_1\right)$$

This comparison reveals that

$$g(\lambda_{CC}) > g(\lambda_{LM}) \quad \text{if} \quad \frac{\sigma_2 - \sigma_1}{\varepsilon - \sigma_2}(1 - \epsilon_1) > \epsilon_1 - \epsilon_2 \quad \text{if } \sigma_2 < \varepsilon \tag{44}$$

$$g(\lambda_{CC}) > g(\lambda_{LM}) \quad \text{if} \quad \frac{\sigma_2 - \sigma_1}{\varepsilon - \sigma_2}(1 - \epsilon_1) < \epsilon_1 - \epsilon_2 \quad \text{if } \sigma_2 > \varepsilon, \tag{45}$$

or condition P1.B in Proposition 1.

It is clear from the graphical analysis (see Figure 1) that for both (CC) and (LM) to hold when (CC) shifts up more than (LM), it is required that λ increases, but κ declines.

Analysis of case 1.b. Recall that in case 1., (LM) is upward-sloping and flatter than (CC), since for any κ, allocating the entire labor endowment to one of the two sectors is never a solution. As k increases, (LM) shifts down and (CC) shifts up. This implies that both κ and λ decline.

Determination of the condition distinguishing 2.b.i. and 2.b.ii. Because in this case, (LM) is downward-sloping, the change in λ and κ is ambiguous. What happens with λ depends on whether (LM) or (CC) shifts left more. To check this, consider a situation where k changes by a proportion $g(k)$, and λ remains unchanged. Then (CC) requires a change in κ of

$$g(\kappa_{CC}) = \frac{\frac{1}{\sigma_2} - \frac{1}{\sigma_1}}{\frac{1}{\sigma_1} + \frac{1}{\sigma_2}\frac{\kappa}{1-\kappa}} g(k) < 0.$$

For (LM) tho hold, a change in κ of

$$g(\kappa_{LM}) = \frac{\left(\frac{1}{\varepsilon} - \frac{1}{\sigma_1}\right)\epsilon_1 - \left(\frac{1}{\varepsilon} - \frac{1}{\sigma_2}\right)\epsilon_2}{\left(\frac{1}{\sigma_1} - \frac{1}{\varepsilon}\right)\epsilon_1 + \left(\frac{1}{\sigma_2} - \frac{1}{\varepsilon}\right)\epsilon_2 \frac{\kappa}{1-\kappa}} g(k)$$

is needed. Under the assumptions that $\sigma_1, \sigma_2 > \varepsilon$ and that condition P1.A does not hold, this is negative. After some tedious algebra, comparing these two expressions reveals that

$$g(\kappa_{LM}) > g(\kappa_{CC}) \quad \text{if} \quad \epsilon_1 - \epsilon_2 > \frac{\sigma_2 - \sigma_1}{\sigma_2 - \varepsilon}\epsilon_1. \tag{P1.C'}$$

This is the condition P1.C in Proposition 1. Since both changes in κ are negative, (CC) shifts left more under this condition. It is clear from the graphical analysis (see Figure 1) that in this case, for both (CC) and (LM) to hold, it is required that λ increases, but κ declines.[31] ■

Proof of Remark 1 to Proposition 1. Consider an increase in k by a proportion $g(k)$. Since $k_1 = \kappa k$, k_1 can only decline if κ falls more than k increases. The decline in κ is bounded by $g(\kappa_{CC})$ or $g(\kappa_{LM})$, whichever declines more. However, these are both smaller in absolute value than $g(k)$. Hence, k_1 grows even if κ falls.

Proof of Proposition 2.

Determination of the condition distinguishing a. and b. (LM) increases in M_1 and M_2 if

$$-\frac{\epsilon_1}{\sigma_1} + \frac{\epsilon_2}{\sigma_2} + \frac{\epsilon_1 - \epsilon_2}{\varepsilon} > 0. \tag{46}$$

Solving this for σ_2/σ_1 shows that this is the case if condition P1.A from Proposition 1 does not hold. Note that the result is the same in cases 1. and 2.

Since (LM) decreases in λ, λ needs to increase after a proportional increase in M_1 and M_2 if condition P1.A does not hold, and decrease if it holds. Hence, if condition P1.A holds, λ needs to decrease following a proportional increase in M_1 and M_2 for (LM) to hold.

Determination of the condition distinguishing a.i. and a.ii. Consider a situation where both M_1 and M_2 change by a proportion $g(M)$, and κ remains unchanged. Then (CC) requires a change in λ of

$$g(\lambda_{CC}) = \frac{\frac{1}{\sigma_2} - \frac{1}{\sigma_1}}{\frac{1}{\sigma_1} + \frac{1}{\sigma_2}\frac{\lambda}{1-\lambda}} g(M) < 0.$$

For (LM) to hold, a change in λ of

$$g(\lambda_{LM}) = \frac{\left(\frac{1}{\sigma_2} - \frac{1}{\varepsilon}\right)\epsilon_2 - \left(\frac{1}{\sigma_1} - \frac{1}{\varepsilon}\right)\epsilon_1}{\frac{1}{\sigma_1}\epsilon_1 + \frac{1}{\sigma_2}\epsilon_2\frac{\lambda}{1-\lambda} + \frac{1}{\varepsilon}\left((1-\epsilon_1) + (1-\epsilon_2)\frac{\lambda}{1-\lambda}\right)} g(M)$$

is needed. Under the assumption that condition P1.A holds, this is negative. After some tedious algebra, comparing these two expressions reveals that

$$g(\lambda_{CC}) > g(\lambda_{LM}) \quad \text{if} \quad \frac{\sigma_2 - \sigma_1}{\varepsilon - \sigma_2}(1-\epsilon_1) < \epsilon_1 - \epsilon_2 \quad \text{if } \sigma_2 < \varepsilon \tag{47}$$

$$g(\lambda_{CC}) > g(\lambda_{LM}) \quad \text{if} \quad \frac{\sigma_2 - \sigma_1}{\varepsilon - \sigma_2}(1-\epsilon_1) > \epsilon_1 - \epsilon_2 \quad \text{if } \sigma_2 > \varepsilon \tag{48}$$

[31] Note that under the conditions of this case, the right hand side of the condition in (P1.C') is larger than the right hand side of the version of P1.A in (P1.A). Hence, knowing whether P1.A holds is not sufficient for knowing whether P1.C holds.

Since both changes in λ are negative, this implies that (CC) shifts down less under this condition. It is clear from the graphical analysis (see Figure 1) that in this case, for both (CC) and (LM) to hold, it is required that λ and κ both decline. If this condition does not hold, (CC) shifts down more, so that for both (CC) and (LM) to hold, it is required that λ declines but κ increases.

Analysis of case 1.b. In this case, (LM) is upward-sloping and shifts up. (CC) shifts down. This implies that both κ and λ increase.

Determination of the condition distinguishing b.i. and b.ii. Consider a situation where M_1 and M_2 change by a proportion $g(M)$, and λ remains unchanged. Then (CC) requires a change in κ of

$$g(\kappa_{CC}) = \frac{\frac{1}{\sigma_1} - \frac{1}{\sigma_2}}{\frac{1}{\sigma_1} + \frac{1}{\sigma_2}\frac{\kappa}{1-\kappa}} \mathrm{d}M > 0.$$

For (LM) to hold, a change in κ of

$$g(\kappa_{LM}) = \frac{\left(\frac{1}{\varepsilon} - \frac{1}{\sigma_1}\right)\epsilon_1 - \left(\frac{1}{\varepsilon} - \frac{1}{\sigma_2}\right)\epsilon_2}{\left(\frac{1}{\varepsilon} - \frac{1}{\sigma_1}\right)\epsilon_1 + \left(\frac{1}{\varepsilon} - \frac{1}{\sigma_2}\right)\epsilon_2\frac{\kappa}{1-\kappa}} g(M)$$

is needed. Under the assumption that condition P1.A does not hold, this is positive. After some tedious algebra, comparing these two expressions reveals that

$$g(\kappa_{LM}) > g(\kappa_{CC}) \quad \text{if} \quad \frac{\sigma_2 - \sigma_1}{\sigma_2 - \varepsilon}\epsilon_1 > \epsilon_1 - \epsilon_2,$$

or if condition P1.C does not hold. Since both changes in κ are positive, this implies that (LM) shifts right more under this condition. It is clear from the graphical analysis (see Figure 1) that in this case, for both (CC) and (LM) to hold, it is required that both κ and λ increase. ■

Proof of Proposition 3. Denote the intensive form of the production function in each sector by $f_s(k_s)$. Using this and denoting the wage-rental ratio by ω, combine (7) and (8) to obtain

$$\omega = \frac{w}{R} = \frac{MPL_s}{MPK_s} = \frac{f_s(k_s; M_s) - k_s f_s'(k_s; M_s)}{f_s'(k_s)} = \frac{f_s}{f_s'} - k_s,$$

i.e.,

$$\frac{f_s'}{f_s} = \frac{1}{\omega + k_s}. \tag{49}$$

Hence

$$\frac{\mathrm{d}\omega}{\mathrm{d}k_s} = -\frac{f_s f_s''}{(f_s')^2} > 0. \tag{50}$$

Then rewrite (7), the condition prescribing that the marginal value product of capital is equated across sectors, as $p_1 f_1'(k_1(\omega), M_1) = p_2 f_2'(k_2(\omega), M_2)$, where the function arguments make explicit that the optimal capital-labor ratio in each sector depends on the input price ratio ω. Rearranging and taking logarithms on both sides, we obtain

$$\ln\frac{p_1}{p_2} = \ln f_2'(k_2(\omega), M_2) - \ln f_1'(k_1(\omega), M_1).$$

Thus

$$\begin{aligned}\frac{\mathrm{d}\ln(p_1/p_2)}{\mathrm{d}\omega} &= \frac{1}{f_2'} f_2'' \frac{\mathrm{d}k_2}{\mathrm{d}\omega} - \frac{1}{f_1'} f_1'' \frac{\mathrm{d}k_1}{\mathrm{d}\omega} = -\frac{f_2'}{f_2} + \frac{f_1'}{f_1} \\ &= -\frac{1}{\omega + k_2} + \frac{1}{\omega + k_1} \\ &= \frac{k_2 - k_1}{(\omega + k_2)(\omega + k_1)} \lesseqgtr 0 \text{ for } \omega \lesseqgtr \tilde{\omega},\end{aligned}$$

where the second equality uses (50) and the third uses (49), and $\bar{\omega}$ denotes the level of the wage-rental ratio implied by $k = \bar{k}$. Given the negative relationship between ω and k, this implies that the relative price of sector 1 output declines in k for k below $\bar{k}$, increases in k for k above $\bar{k}$, and reaches a minimum at $\bar{k}$.

The remaining two claims follow from combining this result with (11). ■

Derivation of Proposition 4. Combining (7) with (10) we reach,

$$\kappa = (1-\kappa)^{1/\sigma_2} \left(\frac{\gamma}{1-\gamma}\right) \left[(1-\alpha)(1-\lambda)^{\frac{\sigma_2-1}{\sigma_2}} \left(\frac{k}{M_2}\right)^{\frac{1-\sigma_2}{\sigma_2}} + \alpha(1-\kappa)^{\frac{\sigma_2-1}{\sigma_2}}\right] \tag{51}$$

Re-arranging (CC) we reach,

$$(k/M_2)^{\frac{1-\sigma_2}{\sigma_2}} = \frac{(1-\lambda)^{\frac{1}{\sigma_2}}}{\lambda} \frac{\kappa}{(1-\kappa)^{1/\sigma_2}} \tag{52}$$

Equation (52) yields the implicit function

$$\lambda = \Lambda(\kappa, k/M_2)$$

with

$$\frac{\partial \Lambda(\kappa, k, M_2)}{\partial \kappa} > 0$$

and

$$\text{Sign}\frac{\partial \Lambda(\kappa, k/M_2)}{\partial (k/M_2)} = \text{Sign}(\sigma_2 - 1)$$

To simplify notation, let $\widetilde{\lambda}$ stand for $\Lambda(\kappa, k/M_2)$. Re-arranging (51)

$$\kappa - (1-\kappa)^{1/\sigma_2}\left(\frac{\gamma}{1-\gamma}\right)\alpha(1-\kappa)^{\frac{\sigma_2-1}{\sigma_2}} = (1-\kappa)^{1/\sigma_2}\left(\frac{\gamma}{1-\gamma}\right)(1-\alpha)(1-\widetilde{\lambda})^{\frac{\sigma_2-1}{\sigma_2}}(k/M_2)^{\frac{1-\sigma_2}{\sigma_2}} \quad (53)$$

Substitute (52) into (53)

$$\frac{\kappa}{(1-\kappa)^{1/\sigma_2}} - \left(\frac{\gamma}{1-\gamma}\right)\alpha(1-\kappa)^{\frac{\sigma_2-1}{\sigma_2}} = \left(\frac{\gamma}{1-\gamma}\right)(1-\alpha)(1-\widetilde{\lambda})^{\frac{\sigma_2-1}{\sigma_2}}\frac{(1-\widetilde{\lambda})^{\frac{1}{\sigma_2}}}{\widetilde{\lambda}}\left(\frac{\kappa}{(1-\kappa)^{1/\sigma_2}}\right)$$

or

$$1 - \left(\frac{\gamma}{1-\gamma}\right)\alpha(1-\kappa)^{\frac{\sigma_2-1}{\sigma_2}}\left(\frac{(1-\kappa)^{1/\sigma_2}}{\kappa}\right) = \left(\frac{\gamma}{1-\gamma}\right)(1-\alpha)\left(\frac{1-\widetilde{\lambda}}{\widetilde{\lambda}}\right)$$

or

$$1 - \left(\frac{\gamma}{1-\gamma}\right)\alpha\left(\frac{1-\kappa}{\kappa}\right) = \left(\frac{\gamma}{1-\gamma}\right)(1-\alpha)\left(\frac{1-\widetilde{\lambda}}{\widetilde{\lambda}}\right)$$

or

$$1 - \left(\frac{\gamma}{1-\gamma}\right)\alpha\left(\frac{1}{\kappa} - 1\right) = \left(\frac{\gamma}{1-\gamma}\right)(1-\alpha)\left(\frac{1}{\widetilde{\lambda}} - 1\right)$$

Thus

$$\frac{\alpha\gamma}{\kappa} = \frac{\widetilde{\lambda} - (1-\gamma)}{\widetilde{\lambda}(1-\gamma)}$$

or equivalently,

$$\widetilde{\lambda} = \frac{\gamma\,(1-\alpha)\,\kappa}{(\kappa - \alpha\gamma)} \equiv \lambda(\kappa, \gamma, \alpha) \quad (54)$$

These two equations equations imply that, *in equilibrium*, $\lambda > (1-\gamma)$ and $\kappa > \alpha\gamma$ in order to satisfy $\kappa > 0$ and $\lambda > 0$ respectively. Also, since $\lambda \leq 1$ and $\kappa \leq 1$, they imply that

$$\kappa > \underline{\kappa} \equiv \frac{\alpha\gamma}{(1-\gamma) + \alpha\gamma} < 1$$

and

$$\lambda > \underline{\lambda} \equiv \frac{\gamma(1-\alpha)}{1-\alpha\gamma} < 1$$

Now, since

$$(\kappa - \alpha\gamma)\,\widetilde{\lambda} - \gamma\,(1-\alpha)\,\kappa = 0$$

let us define

$$\Omega(\kappa, k/M_2) \equiv (\kappa - \alpha\gamma)\,\Lambda(\kappa, k/M_2) - \gamma\,(1-\alpha)\,\kappa = 0$$

This equation yields κ as an implicit function of k and M_2, which we denote as $\kappa = \kappa^*(k/M_2)$. Then

$$\frac{d\kappa^*}{d(k/M_2)} = -\frac{\frac{\partial \Omega}{\partial (k/M_2)}}{\frac{\partial \Omega}{\partial \kappa}}$$

where

$$\frac{\partial \Omega}{\partial \kappa} = [\Lambda(\kappa, k/M_2) - \gamma\,(1-\alpha)] + (\kappa - \alpha\gamma)\,\frac{\partial \Lambda(\kappa, k/M_2)}{\partial \kappa} > 0$$

and

$$\frac{\partial \Omega}{\partial (k/M_2)} = (\kappa - \alpha\gamma)\,\frac{\partial \Lambda(\kappa, k/M_2)}{\partial (k/M_2)} > 0 \text{ iff } \sigma_2 > 1$$

Therefore $\frac{d\kappa^*}{d(k/M_2)} < 0$ iff $\sigma_2 > 1$. Finally,

$$\lambda^*(k/M_2) = \Lambda(\kappa^*(k/M_2), k/M_2) = \frac{\gamma\,(1-\alpha)\,\kappa^*(k/M_2)}{\kappa^*(k/M_2) - \alpha\gamma} \tag{55}$$

Thus

$$\frac{d\lambda^*}{d(k/M_2)} = -\left(\frac{\alpha}{1-\alpha}\right)\left(\frac{\lambda^*}{\kappa^*}\right)^2 \frac{d\kappa^*}{d(k/M_2)} \tag{56}$$

This equation shows that $\lambda^*(k/M_2)$ and $\kappa^*(k/M_2)$ always move in opposite directions as k/M_2 increases (even though $\partial\Lambda(\kappa, k/M_2)/\partial\kappa$, i.e., the slope of the contract curve, for a given k/M_2, is always positive).

Combining (52) and (54) and taking logs we reach,

$$\frac{1-\sigma_2}{\sigma_2}\ln\left(\frac{k}{M_2}\right) = \frac{1}{\sigma_2}\ln(1-\lambda(\kappa,\alpha,\gamma)) - \ln\lambda(\kappa,\alpha,\gamma) + \ln\kappa - \frac{1}{\sigma_2}\ln(1-\kappa)$$

This relationship is monotone decreasing (iff $\sigma_2 > 1$): an increase in $\frac{k}{M_2}$ leads to a fall in κ :

$$\left(\frac{1-\sigma_2}{\sigma_2}\right)\left(\frac{dk}{k} - \frac{dM_2}{M_2}\right) = G\,(\kappa)\,d\kappa \tag{57}$$

where

$$G(\kappa) \equiv \left[\frac{1}{\sigma_2\,(1-\lambda(\kappa,\alpha,\gamma))} + \frac{1}{\lambda(\kappa,\alpha,\gamma)}\right]\left(\frac{\lambda(\kappa,\alpha,\gamma)}{\kappa}\right)^2\left(\frac{\alpha}{1-\alpha}\right) + \left[\frac{1}{\kappa} + \frac{1}{\sigma_2(1-\kappa)}\right] > 0$$

where $\lambda(\kappa,\alpha,\gamma)$ is given by (54). Combining equations (56) and (57) we reach the results summarized in this Proposition. ■

Proof of Proposition 5. Log-differentiating (15) and using (3) we reach,

$$\hat{\chi} = \hat{K} - n - m_1 = s\frac{Y}{K} - (\delta + n + m_1) \tag{58}$$

We must find an expression for $\frac{Y}{K}$ in terms of our two endogenous variables, χ and κ. Notice that (7) together with (10) imply,

$$\left(\frac{Y_1}{Y_2}\right)^{\frac{1-\varepsilon}{\varepsilon}} = \left(\frac{1-\kappa}{\kappa}\right)\left(\frac{\gamma}{1-\gamma}\right)\frac{\alpha_1}{\alpha_2} \tag{59}$$

and dividing both sides of (1) by Y_1 we have,

$$\frac{Y}{Y_1} = \left[\gamma + (1-\gamma)\left(\frac{Y_2}{Y_1}\right)^{\frac{\varepsilon-1}{\varepsilon}}\right]^{\frac{\varepsilon}{\varepsilon-1}}$$

Combining these two expressions we reach,

$$\frac{Y}{Y_1} = \gamma^{\frac{\varepsilon}{\varepsilon-1}}\left[1+\frac{\alpha_1}{\alpha_2}\left(\frac{1-\kappa}{\kappa}\right)\right]^{\frac{\varepsilon}{\varepsilon-1}} \equiv \eta$$

Then,

$$Y = \eta Y_1 = \eta\left(M_1\lambda L\right)^{1-\alpha_1}(\kappa K)^{\alpha_1} = \eta\lambda^{1-\alpha_1}\kappa^{\alpha_1}\chi^{\alpha_1-1}K \tag{60}$$

which, when combined with (58), yields (17).

Now using equation (59) we reach

$$\frac{\dot{\kappa}}{\kappa} = -\left(\frac{1-\varepsilon}{\varepsilon}\right)(1-\kappa)\widehat{\left(\frac{Y_1}{Y_2}\right)} \tag{61}$$

Since $\frac{Y_1}{Y_2} = \lambda^{1-\alpha_1}(1-\lambda)^{-(1-\alpha_2)}\kappa^{\alpha_1}(1-\kappa)^{-\alpha_2}\left(\frac{L}{K}\right)^{\alpha_2-\alpha_1}\left(\frac{M_1}{M_2}\right)^{\alpha_2-\alpha_1}$, we obtain

$$\widehat{\left(\frac{Y_1}{Y_2}\right)} = \left(\frac{(1-\alpha_1)}{\lambda}+\frac{(1-\alpha_2)}{(1-\lambda)}\right)\frac{\partial\lambda}{\partial\kappa}\dot{\kappa} + \alpha_1\frac{\dot{\kappa}}{\kappa} + \alpha_2\frac{\dot{\kappa}}{(1-\kappa)} + (\alpha_1-\alpha_2)\left(\hat{\chi}+m_2\right) \tag{62}$$

Substituting (62) into equation (61), using $\frac{\partial\lambda}{\partial\kappa} = \frac{\alpha_1}{\alpha_2}\frac{(1-\alpha_2)}{(1-\alpha_1)}\left(\frac{\lambda}{\kappa}\right)^2$ and $\frac{\kappa}{\lambda} = \left(\frac{1-\kappa}{1-\lambda}\right)\left(\frac{\alpha_1}{\alpha_2}\right)\left(\frac{1-\alpha_2}{1-\alpha_1}\right)$, we arrive at

$$\dot{\kappa} = -\left(\frac{1-\varepsilon}{\varepsilon}\right)\left[\dot{\kappa}\left(1+(\alpha_1-\alpha_2)(\lambda-\kappa)\right)+\kappa(1-\kappa)(\alpha_1-\alpha_2)\left(\hat{\chi}+m_2\right)\right]$$

which after some manipulation yields (18). ■

Proof of Theorem 1. Consider the curve $\widehat{\chi}=$ constant. Differentiating (17) (keeping $\widehat{\chi}=$ constant) we reach a positive relationship between χ and κ along the $\widehat{\chi}=$ constant schedule,

$$\frac{\partial \kappa}{\partial \chi} = \frac{(1-\alpha_1)}{\chi\left(\frac{\frac{\partial \eta}{\partial \kappa}}{\eta} + (1-\alpha_1)\frac{\frac{\partial \lambda}{\partial \kappa}}{\lambda} + \frac{\alpha_1}{\kappa}\right)}$$
$$= \frac{\kappa^2 (1-\alpha_1)}{\chi\left(\frac{\varepsilon}{1-\varepsilon}\gamma\eta^{\frac{1}{\varepsilon}}\frac{\alpha_1}{\alpha_2} + (1-\alpha_2)\frac{\alpha_1}{\alpha_2}\lambda + \alpha_1\kappa\right)} > 0 \tag{63}$$

Thus this schedule is upward sloping in the space (χ, κ), where χ is measured along the horizontal axis. The equation for the curve $\widehat{\chi} = b$ (where b is any constant such that $\delta + n + m_1 - b > 0$) is

$$\chi^{\frac{1}{1-\alpha_1}} = \frac{\kappa^{\alpha_1}}{\left[1 + \frac{\alpha_1}{\alpha_2}\left(\frac{1-\alpha_2}{1-\alpha_1}\right)\frac{1-\kappa}{\kappa}\right]^{1-\alpha_1}}\left[\frac{1}{\left(1 + \frac{\alpha_2}{\alpha_1}\frac{1-\kappa}{\kappa}\right)^{\frac{\varepsilon}{1-\varepsilon}}}\right]\frac{s\gamma^{\frac{\varepsilon}{\varepsilon-1}}}{\delta + n + m_1 - b}$$

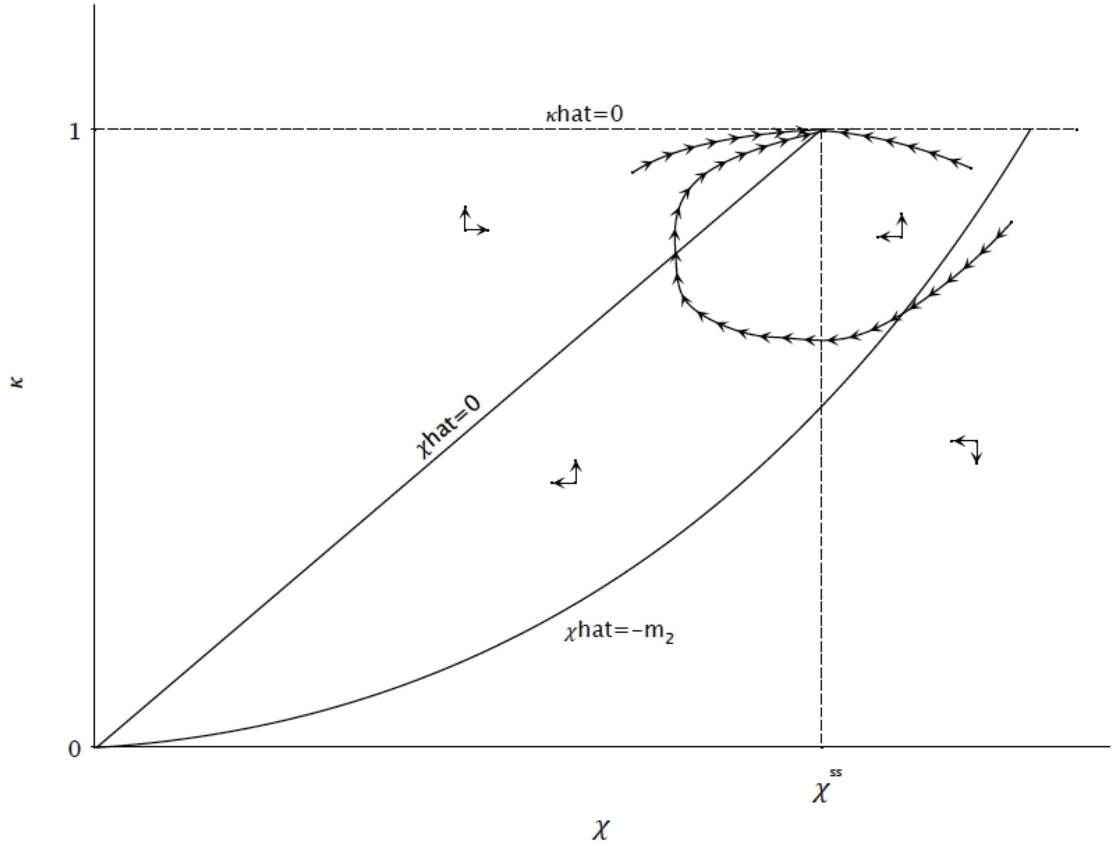
In particular, the curve $\widehat{\chi} = 0$ meets the line $\kappa = 1$ at $\chi^{ss} = \left(\frac{s\gamma^{\frac{\varepsilon}{\varepsilon-1}}}{\delta + n + m_1}\right)^{\frac{1}{1-\alpha_1}}$. Since $\frac{\partial \widehat{\chi}}{\partial \chi} < 0$, capital per unit of effective labor, χ, increases below (to the left) the $\hat{\chi} =$ 0-line. its steady state value. Now we turn to equation (18), setting $\widehat{\kappa} = 0$ requires $\kappa(1-\kappa)(\alpha_2 - \alpha_1)(\hat{\chi} + m_2) = 0$ which implies either $\kappa = 1$ or $\hat{\chi} = -m_2$. The latter case cannot be a steady state. Therefore the only non-trivial steady state allocation of capital is $\kappa^{ss} = 1$, i.e. the relevant $\widehat{\kappa} = 0$-line that determines the steady state is the horizontal line $\kappa = 1$. Below the $\widehat{\kappa} = 0$-line and above the locus where $\hat{\chi} = -m_2$, it is clear that κ is growing since $\alpha_2 > \alpha_1$, $\lambda > \kappa$ and $\widehat{\chi} - m_2 > 0$. Below the locus where $\hat{\chi} = -m_2$ a similar reasoning implies that κ is decreasing. Figure 6 illustrates the phase diagram associated with this dynamical system.[32] It is clear that κ and χ asymptotically reach their steady state levels given by (19).

The growth rates of the asymptotically dominant sector (and therefore of the overall economy) are derived by combining the steady state solutions and the growth rates of the exogenous variables with (6), (15), and (60). The growth rates for sector 2 are given by the solution of the system of three equations on g_2^{ss}, z_2^{ss} and n_2^{ss} that results from the log-differentiation of (4), (7) and (8). ■

Proof of Proposition 6. Using (58) we need an expression for $\frac{Y}{K}$ in terms of our two

[32]Notice that there might be several values of $\kappa(0)$ for a given $\chi(0)$ since $\kappa(0) = \mathcal{K}(\chi(0), A(0))$.

Figure 6: Phase Diagram



endogenous variables. Under the NP restrictions, equation (1) can be written as,

$$\begin{aligned} Y &= \left[\gamma Y_1^{\frac{\varepsilon-1}{\varepsilon}} + (1-\gamma) Y_2^{\frac{\varepsilon-1}{\varepsilon}}\right]^{\frac{\varepsilon}{\varepsilon-1}} \\ &= \left[\gamma \left((M_1 \lambda L)^{1-\alpha} (\kappa K)^{\alpha}\right)^{\frac{\varepsilon-1}{\varepsilon}} + (1-\gamma)\left((M_2 (1-\lambda) L)^{1-\alpha} ((1-\kappa) K)^{\alpha}\right)^{\frac{\varepsilon-1}{\varepsilon}}\right]^{\frac{\varepsilon}{\varepsilon-1}} \\ &= L^{1-\alpha} K^{\alpha} \left[\gamma \left(M_1^{1-\alpha} \kappa\right)^{\frac{\varepsilon-1}{\varepsilon}} + (1-\gamma)\left({M_2}^{1-\alpha}(1-\kappa)\right)^{\frac{\varepsilon-1}{\varepsilon}}\right]^{\frac{\varepsilon}{\varepsilon-1}} \end{aligned}$$

where we have used the fact that the capital-labor ratios are identical across sectors, implying $\kappa = \lambda$. The preceding equation gives

$$\begin{aligned} \frac{Y}{K} &= \left(\frac{K}{LM_1}\right)^{\alpha-1} M_1^{\alpha-1} \left[\gamma \left(M_1^{1-\alpha} \kappa\right)^{\frac{\varepsilon-1}{\varepsilon}} + (1-\gamma)\left({M_2}^{1-\alpha}(1-\kappa)\right)^{\frac{\varepsilon-1}{\varepsilon}}\right]^{\frac{\varepsilon}{\varepsilon-1}} \\ &= \chi^{\alpha-1}\left[\gamma \kappa^{\frac{\varepsilon-1}{\varepsilon}} + (1-\gamma)\left(\left(\frac{M_2}{M_1}\right)^{1-\alpha}(1-\kappa)\right)^{\frac{\varepsilon-1}{\varepsilon}}\right]^{\frac{\varepsilon}{\varepsilon-1}} \equiv \xi \chi^{\alpha-1} \end{aligned}$$

from which we obtain (20). Combining the (LM) curve with the fact that the (CC) curve implies $\lambda = \kappa$, the derivation of (21) is straight forward. ■

Proof of Proposition 7. Using (15) we can rewrite the CC curve as,

$$\frac{(1-\kappa)^{\frac{1}{\sigma_2}}}{\kappa}\frac{\lambda}{(1-\lambda)^{\frac{1}{\sigma_2}}}=\chi^{\frac{\sigma_2-1}{\sigma_2}}$$

which, after using (55) to replace λ, becomes

$$\chi=(\gamma(1-\alpha))^{\frac{\sigma_2}{\sigma_2-1}}\frac{(1-\kappa)^{\frac{1}{\sigma_2-1}}}{(\kappa-\alpha\gamma)\left(\kappa\left(1-\gamma(1-\alpha)\right)-\alpha\gamma\right)^{\frac{1}{\sigma_2-1}}}$$

where $\chi'(\kappa)<0$ (resp. $\chi'(\kappa)>0$) for all $\kappa\in(\underline{\kappa},1)$ if $\sigma_2>1$ (resp. $\sigma_2<1$). Furthermore, it is worth noticing that when $\sigma_2>1$, $\chi(1)=0$ and $\lim_{\kappa\to\underline{\kappa}}\chi(\kappa)=\infty$, and when $\sigma_2<1$, $\chi(\underline{\kappa})=0$ and $\lim_{\kappa\to1}\chi(\kappa)=\infty$.

Given (15) and $M_1=M_2$,the rate of change of the normalized capital stock is,

$$\widehat{\chi}=\widehat{K}-\widehat{M_1}-\widehat{L}=s\frac{Y}{K}-\delta-m-n$$

Since $\varepsilon=1$ and sector 1 has the Cobb-Douglas technology while sector 2 has the CES technology, the agregate output-capital ratio is given by

$$\frac{Y}{K}=\left[(\lambda\chi^{-1})^{1-\alpha}(\kappa)^{\alpha}\right]^{\gamma}\left[(1-\alpha)\left((1-\lambda)\chi^{-1}\right)^{\frac{\sigma_2-1}{\sigma_2}}+\alpha((1-\kappa))^{\frac{\sigma_2-1}{\sigma_2}}\right]^{\frac{(1-\gamma)\sigma_2}{\sigma_2-1}}$$

which can be expressed using (59), (55), and (23) as

$$\frac{Y}{K}=D\pi(\kappa) \tag{64}$$

where D and $\pi(\kappa)$ are defined by (25) and (26) respectively. Notice that when $\sigma_2>1$ (respectively, $\gamma<\sigma_2<1$), $\pi(\kappa)$ is an increasing resp. decreasing) function defined over the interval $[\underline{\kappa},1]$, with $\pi(\underline{\kappa})=0$ (resp. $\pi(\kappa)=\infty$) and $\lim_{\kappa\to1}\pi(\kappa)=\infty$ (resp. $\pi(1)=0$).

Therefore

$$\widehat{\chi}=sD\pi(\kappa)-(\delta+m+n) \tag{65}$$

Finally log-differentiating (23)

$$\widehat{\chi}=-H(\kappa)\dot{\kappa} \tag{66}$$

where $H(\kappa)$ is defined by

$$H(\kappa)\equiv-\left(\frac{1}{\sigma_2-1}\right)\frac{1}{1-\kappa}-\frac{(\sigma_2-1)}{(\kappa-\alpha\gamma)(\sigma_2-1)}-\left(\frac{1}{\sigma_2-1}\right)\frac{1-\gamma(1-\alpha)}{\kappa(1-\gamma(1-\alpha))-\alpha\gamma}$$

Then
$$H(\kappa) = -\left(\frac{1}{\sigma_2 - 1}\right)\left[\frac{1}{1-\kappa} + \frac{\sigma_2 - 1}{(\kappa - \alpha\gamma)} + \frac{1-\gamma(1-\alpha)}{\kappa(1-\gamma(1-\alpha)) - \alpha\gamma}\right]$$
where the terms inside [..] is equal to
$$\frac{(1-\gamma+\alpha\gamma)(1-\kappa)(\kappa-\alpha\gamma) + \sigma_2(1-\kappa)(\kappa(1-\gamma(1-\alpha)) - \alpha\gamma)}{(1-\kappa)(\kappa-\alpha\gamma)(\kappa(1-\gamma(1-\alpha)) - \alpha\gamma)}$$
i.e.
$$\frac{(1-\gamma+\alpha\gamma)(\kappa-\alpha\gamma) + \sigma_2(\kappa(1-\gamma(1-\alpha)) - \alpha\gamma)}{(\kappa-\alpha\gamma)(\kappa(1-\gamma(1-\alpha)) - \alpha\gamma)}$$
which is positive for all $\kappa \in [\underline{\kappa}, 1]$.

Combining (66) with (65) yields (24). ■

Proof of Lemma 2. This proof follows Jones (1965) and Miyagiwa and Papageourgiou (2007). The dual relationship between sectoral prices and input prices and factor endowments and sectoral outputs are given by,

$$C_1(w,R) \equiv \frac{L_1}{Y_1}w + \frac{K_1}{Y_1}R = p_1 \tag{67}$$
$$C_2(w,R) \equiv \frac{L_2}{Y_2}w + \frac{K_2}{Y_2}R = p_2 \tag{68}$$
$$Y_1C_{1w} + Y_2C_{2w} = L \tag{69}$$
$$Y_1C_{1R} + Y_2C_{2R} = K \tag{70}$$

where $C_i(w,R)$ is the unit cost function for sector $s = 1,2$ and C_{ij} are its partial derivatives with respect to each factor price $j = w, R$.

Differentiating the previous expressions we reach the following relationships,

$$\frac{wL_1}{p_1Y_1}\hat{w} + \frac{RK_1}{p_1Y_1}\hat{R} = (1-\epsilon_1)\hat{w} + \epsilon_1\hat{R} = \widehat{p}_1 \tag{71}$$
$$\frac{wL_2}{p_2Y_2}\hat{w} + \frac{RK_2}{p_2Y_2}\hat{R} = (1-\epsilon_2)\hat{w} + \epsilon_2\hat{R} = \widehat{p}_2 \tag{72}$$
$$\lambda\left(\hat{Y}_1 + \widehat{C}_{1w}\right) + (1-\lambda)\left(\hat{Y}_2 + \widehat{C}_{2w}\right) = \hat{L} \tag{73}$$
$$\kappa\left(\hat{Y}_1 + \widehat{C}_{1R}\right) + (1-\kappa)\left(\hat{Y}_2 + \widehat{C}_{2R}\right) = \hat{K} \tag{74}$$

where ϵ_s is the capital income share in sector s and we use the fact that sectoral production functions are homogeneous of degree one.

Subtracting (71) and (72),

$$(\epsilon_2 - \epsilon_1)\left(\hat{w} - \hat{R}\right) = \widehat{p}_1 - \widehat{p}_2 \tag{75}$$

Using the definition of the sector-specific elasticity of substitution, $\sigma_s \equiv \frac{C_s C_{swR}}{C_{sw} C_{sR}}$, since we can express the factor income shares as $\epsilon_s = \frac{rC_{sR}}{C_s}$ and $1-\epsilon_s = \frac{wC_{sw}}{C_s}$, we reach the following rates of change of partial derivatives of the unit cost functions,

$$\widehat{C}_{sw} = \frac{C_{sww}dw + C_{swr}dR}{C_{sw}} = \frac{-C_{swR}\frac{R}{w}dw + C_{swR}dR}{C_{sw}} =$$
$$-\frac{(C_{swR}R\hat{w} - C_{swR}dR)}{C_{sw}} = -\frac{C_{swR}R}{C_{sw}}\left(\hat{w} - \hat{R}\right) = -\frac{C_s C_{swR}}{C_{sw} C_{sR}}\frac{RC_{sR}}{C_s}\left(\hat{w} - \hat{R}\right)$$

where the second equality uses the fact that $C_{sw}\left(w, \hat{R}\right)$ is homogeneous of degree 0. As a result

$$\widehat{C}_{sw} = -\sigma_s \epsilon_s \left(\hat{w} - \hat{R}\right) \tag{76}$$

$$\widehat{C}_{sR} = \sigma_s (1 - \epsilon_s)\left(\hat{w} - \hat{R}\right) \tag{77}$$

Replacing (76) and (77) in (73) and (74) and subtracting them we reach,

$$(\lambda - \kappa)(\hat{Y}_1 - \hat{Y}_2) = \left(\hat{L} - \hat{K}\right) + \Theta\left(\hat{w} - \hat{R}\right) \tag{78}$$

where $\Theta \equiv \lambda\sigma_1\epsilon_1 + (1-\lambda)\sigma_2\epsilon_2 + \kappa\sigma_1(1-\epsilon_1) + (1-\kappa)\sigma_2(1-\epsilon_2)$.

Finally, we use (11) to reach

$$\hat{Y}_1 - \hat{Y}_2 = -\varepsilon\left(\widehat{p}_1 - \widehat{p}_2\right) \tag{79}$$

Since the aggregate elasticity of substitution is defined as $\sigma \equiv -\frac{\left(\hat{L} - \hat{K}\right)}{\left(\hat{w} - \hat{R}\right)}$ we combine (75) and (78) in (79) to reach (41).

Appendix B

In order to show that $f(\chi)$ is a strictly concave and increasing function of χ with $f(0) = 0$, we show the existence of an aggregate production $Y = F(K, L)$ that inherits the properties of the sectoral production functions.

Assumption P1: There are two intermediate goods, produced by capital and labor. The production function for intermediate good i is denoted by $Y_i = g^i(K_i, L_i)$, where $g^i : R_+^2 \to R_+$ is linear homogeneous, increasing in each argument, and strictly quasi-concave, with $g^i(0,0) = 0$. In particular, for all $(K_i', L_i'), (K_i'', L_i'') \in \mathbb{R}_{++}^2$, if $(\overline{K}_i, \overline{L}_i) \equiv \alpha(K_i', L_i') + (1-\alpha)(K_i'', L_i'')$, for some $\alpha \in (0,1)$ then $g^i(\overline{K}_i, \overline{L}_i) > \min\{g^i(K_i', L_i'), g^i(K_i'', L_i'')\}$.

Define the set

$$S_i = \left\{(Y_i, K_i, L_i) \in \mathbb{R}_+^3 | Y_i \leq g^i(K_i, L_i)\right\}$$

Clearly, this set is a convex cone with vertex $(0,0,0)$: if $x \equiv (Y_i, K_i, L_i) \in S_i$ then $\beta x \in S_i$ for all $\beta \geq 0$; and $x', x'' \in S_i \implies \alpha x' + (1-\alpha)x'' \in S_i = S_i$.

Assumption P2: The production function of the final good is $Y = H(Y_1, Y_2)$, where $H : \mathbb{R}_+^2 \to \mathbb{R}_+$ linear homogeneous, increasing in each argument, and strictly quasi-concave, with $H(0,0) = 0$. For all $(Y_1', Y_2'), (Y_1'', Y_2'') \in R_{++}^2$, if $(\overline{Y}_1, \overline{Y}_2) = \alpha(Y_1', Y_2') + (1-\alpha)(Y_1'', Y_2'')$ for some $\alpha \in (0,1)$, then $H(\overline{Y}_1, \overline{Y}_2) > \min\{H(Y_1', Y_2'), H(Y_1'', Y_2'')\}$.

Define

$$S = \left\{(Y, Y_1, Y_2) \in \mathbb{R}_+^3 | Y \leq H(Y_1, Y_2)\right\}$$

This set is a convex cone with vertex $(0,0,0)$.

In order to define the aggregate production function, consider the set

$$Z = \left\{(Y, K, L) \in \mathbb{R}_+^3 | \exists (Y_1, Y_2, K_1, K - K_1, L_1, L - L_1) \in \mathbb{R}_+^6, (Y, Y_1, Y_2) \in S, (Y_i, K_i, L_i) \in S_i\right\}$$

It is straightforward to show that Z is a convex cone with vertex $(0,0,0)$.

For any vector $(K, L) \in \mathbb{R}_+^2$, let us define

$$F(K, L) = \sup_Y \{Y \in \mathbb{R}_+ | (Y, K, L) \in Z\}$$

Then $F(.,.)$ is the aggregate production function. It is easy to see that $F(.,.)$ is homogeneous of degree 1, and strictly quasi-concave. For all $(K', L'), (K'', L'') \in \mathbb{R}_{++}^2$, if $(\overline{K}, \overline{L}) = \alpha(K', L') + (1-\alpha)(K'', L'')$ for some $\alpha \in (0,1)$, then $F(\overline{K}, \overline{L}) > \min\{F(K', L'), F(K'', L'')\}$.

Now, we define aggregate output per unit of effective labor as

$$\frac{Y}{ML} = F\left(\frac{K}{ML}, 1\right) \equiv f(\chi) \tag{80}$$

Clearly $f(\chi)$ is strictly increasing, because $F(.,.)$ is homogeneous of degree 1 and $F(a,b) > 0$ for $(a,b) > (0,0)$. Under the specific functional forms assumed Section 2.2.3, a direct verification of the properties of $f(\chi)$ is given below.

First, we verify that $f(\chi)$ is strictly increasing. From (23) we know that there is a monotone decreasing (increasing) relationship between κ and χ if $\sigma_2 > 1$ ($\sigma_2 < 1$). So we need to show that $R(\kappa)\chi(\kappa)$ is decreasing (increasing) in κ if $\sigma_2 > 1$ ($\sigma_2 < 1$).

Combining (26) and (23) we have that,

$$\pi(\kappa(\chi))\chi = \frac{(\gamma(1-\alpha))^{\frac{\sigma_2}{\sigma_2-1}}(1-\kappa)^{\frac{\alpha\gamma}{\sigma_2-1}}\kappa^{(\sigma_2-\gamma)/(\sigma_2-1)}}{(\kappa-\alpha\gamma)(\kappa(1-\gamma(1-\alpha))-\alpha\gamma)^{\frac{1-\gamma(1-\alpha)}{\sigma_2-1}}} \equiv h(\kappa) > 0 \text{ for } \kappa \in (\underline{\kappa}, 1) \tag{81}$$

Then we can express (80) as

$$f(\chi) = Dh(\kappa(\chi))$$

and therefore

$$f'(\chi) = Dh'(\kappa)\frac{d\kappa}{d\chi} = Dh(\kappa)\frac{d\ln h}{d\kappa}\frac{d\kappa}{d\chi} = \frac{Dh(\kappa)}{\left(\frac{d\chi}{d\kappa}\right)}\frac{d\ln h}{d\kappa} = D\frac{h(\kappa)}{\chi}\frac{\frac{d\ln h}{d\kappa}}{\frac{d\ln\chi}{d\kappa}} = D\pi(\kappa)\frac{\frac{d\ln h}{d\kappa}}{\frac{d\ln\chi}{d\kappa}} \tag{82}$$

Log-differentiating (81) we reach

$$\frac{d\ln h}{d\kappa} = -\frac{\alpha\gamma}{(\sigma_2-1)(1-\kappa)} - \frac{[1-\gamma(1-\alpha)]^2}{(\sigma_2-1)(\kappa(1-\gamma(1-\alpha))-\alpha\gamma)} + \frac{\sigma_2-\gamma}{(\sigma_2-1)\kappa} - \frac{1}{\kappa-\alpha\gamma} \tag{83}$$

that after re-arranging becomes

$$\frac{d\ln h}{d\kappa} = \frac{1}{(1-\sigma_2)V(\kappa)}\left(\xi(\kappa) + \alpha\gamma\sigma_2(1-\kappa)(\kappa(1-\gamma(1-\alpha))-\alpha\gamma)\right) \tag{84}$$

where

$$\xi(\kappa) \equiv \alpha\gamma\left[(1-\gamma+\alpha\gamma)\kappa^2 - 2\alpha\kappa\gamma + \alpha\gamma^2\right]$$

and

$$V(\kappa) \equiv (1-\kappa)\kappa(\kappa-\alpha\gamma)(\kappa(1-(1-\alpha)\gamma)-\alpha\gamma) > 0 \text{ for all } \kappa \in (\underline{\kappa}, 1)$$

Notice that $\xi(.)$ is a strictly convex function that attains a minimum at $\kappa = \dfrac{\alpha\gamma}{1-\gamma+\alpha\gamma} = \underline{\kappa}$. Furthermore evaluated at $\underline{\kappa}$ we find $\xi(\underline{\kappa}) > 0$, so it follows that $\xi(\kappa) \geq \xi(\underline{\kappa}) > 0$ for all $\kappa \in [\underline{\kappa}, 1]$. Therefore, we conclude that $sign\left(\dfrac{d\ln h}{d\kappa}\right) = sign(1-\sigma_2)$■

The function $f(\chi)$ displays the following properties: $f(0) = 0$ and $f(\infty) = \infty$.

Proof: Consider the case $\sigma_2 > 1$. As $\chi \to 0$, $\kappa(\chi) \to 1$, and since $\lim_{\kappa\to 1} h(\kappa) = 0$, we have $f(0) = 0$. As $\chi \to \infty$, $\kappa(\chi) \to \underline{\kappa}$ and since $\lim_{\kappa\to\underline{\kappa}} h(\kappa) = \infty$, we have $f(\infty) = \infty$. A similar reasoning can be applied when $\sigma_2 < 1$.■

Finally, $f(.)$ is strictly concave in χ because $F(K,L)$ is strictly increasing, homogeneous of degree 1, and strictly quasiconcave.

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