DISCUSSION PAPER SERIES

IZA DP No. 692

Spatial Mismatch, Search Effort and Urban Spatial Structure

Tony E. Smith
Yves Zenou

January 2003

Forschungsinstitut
zur Zukunft der Arbeit
Institute for the Study
of Labor

Spatial Mismatch, Search Effort and Urban Spatial Structure

Tony E. Smith
University of Pennsylvania

Yves Zenou
University of Southampton, GAINS, CEPR and IZA Bonn

Discussion Paper No. 692
January 2003

IZA

P.O. Box 7240
D-53072 Bonn
Germany

Tel.: +49-228-3894-0
Fax: +49-228-3894-210
Email: iza@iza.org

IZA Discussion Paper No. 692
January 2003

ABSTRACT

Spatial Mismatch, Search Effort and Urban Spatial Structure[*]

The aim of this paper is to provide a new mechanism for the spatial mismatch hypothesis. Spatial mismatch can here be the result of optimizing behavior on the part of the labor market participants. In particular, the unemployed can choose low amounts of search and long-term unemployment if they reside far away from jobs. They choose voluntary not to relocate close to jobs because the short-run gains (low land rent and large housing consumption) are big enough compared to the long-run gains of residing near jobs (higher probability of finding a job).

JEL Classification: D83, J64, R14

Keywords: job matching, search intensities, urban segregation, MTO

Corresponding author:

Yves Zenou
Department of Economics
University of Southampton
SO17 1BJ
Southampton
U.K.
Tel.: +44 23 80 59 32 64
Fax: +44 23 80 59 38 58
Email: yz@soton.ac.uk

[*] The authors would like to thank the Editor, Jan Brueckner, and two anonymous referees for helpful comments.

1. Introduction

The spatial mismatch hypothesis, first formulated by Kain [20], states that, residing in urban segregated areas distant from and poorly connected to major centers of employment growth, black workers face strong geographic barriers to finding and keeping well-paid jobs. In the U.S. context, where jobs have been decentralized and blacks have stayed in the central part of cities, the main conclusion of the spatial mismatch hypothesis is to put forward the distance to jobs as the main culprit for the high unemployment rates among blacks.

Since the study of Kain, dozens of empirical studies have been carried out trying to test this hypothesis (see the surveys by Holzer [16], Kain [21] and Ihlanfeldt and Sjoquist [18]). The usual approach is to relate a measure of labor-market outcomes, based on either individual or aggregate data, to another measure of job access, typically some index that captures the distance from residences to centers of employment. The weight of the evidence suggests that bad job access indeed worsens labor-market outcomes, confirming the spatial mismatch hypothesis.

The theoretical foundations behind these empirical results remain however unclear. If researchers do agree on the causes (housing discrimination, social interactions) and on the consequences of the spatial mismatch hypothesis (higher unemployment rates and lower wages for black workers), the economic mechanisms and thus the policy implications are difficult to identify.

A first theoretical view developed by Brueckner and Martin [8] and Brueckner and Zenou [9] is to argue that suburban housing discrimination skews black workers towards the Central Business District (CBD) and thus keeps black residences remote from the suburbs. Since black workers who work in the Suburban Business District (SBD) support longer and more costly commuting costs, few of them will accept SBD jobs. As a result, the black CBD labor pool is large relative to the SBD pool, which exerts a strong pressure for central jobs. If wages are set to deter shirking (efficiency wages), it is easy to see that unemployment rates are higher and wages lower in the CBD than in the SBD since unemployment acts as a worker discipline device so that high unemployment rates are associated with low wages. The strength of this argument is that it works with a simple minimum wage model since, because of restricted mobility, the CBD-labor supply is much more higher than the SBD-labor supply.[1] The

[1] Zax and Kain [43] have studied the case of a large firm in the service industry which relocated from the center of Detroit to the suburb Dearborn in 1974. Among workers whose commuting time was increased, black workers were over-represented, and not all could follow

main welfare recommendations of this model (which is also the common view of the spatial literature; see, in particular, Ihlanfeldt and Sjoquist [18] and Pugh [30]) is clearly through transportation solutions since they ameliorate job access. The government can either improve the transportation network or directly subsidize blacks' commuting costs.[2]

Wasmer and Zenou [41] have proposed a different theory for the spatial mismatch hypothesis. Using a search-matching model, they state that residential distance to jobs prevents black workers to obtain information about jobs and thus isolate them from employment centers. Indeed, if one believes that information decreases with distance to jobs, then restricted mobility (due for example to housing discrimination) can have a dramatic impact on the labor market. Little information reaches the area where blacks live and, as a result, lowers their search efficiency and thus their probability to find a job. The key question is then the negative relationship between distance to jobs and information. If firms advertize locally jobs (for example place help-wanted signs in their windows or place ads in *local* newspapers), then, obviously, workers living further away from these firms have less information on these jobs than those residing closer. This view has empirical supports. For example, Turner [39] have shown that, in Detroit, suburban firms using local recruitment methods (such as local newspapers or help-wanted signs in their windows) had few inner-city black applicants whereas those using general formal methods (such as city newspapers) had much more inner-city applicants. Holzer and Reaser [17] have also found that, in four major metropolitan areas (Atlanta, Boston, Detroit and Los Angeles), inner-city black workers apply less frequently for jobs in the suburbs than in central cities because of higher costs of applying and/or lower information flows. The policy implication of this model is thus quite different that the previous one. The government should ameliorate the information about jobs and thus the search efficiency of inner-city black residents (for example, better information, better market structure organization).

Another theoretical view has been proposed by Coulson, Laing and Wang [11]. Using also a search-matching model, they assume that the fixed entry cost of firms is greater in the CBD than in the SBD and that workers

the firm. This had two consequences: first, segregation forced some blacks to quit their jobs. Second, the share of black workers applying for jobs to the firm drastically decreased (53% to 25% in 5 years before and after the relocation), and the share of black workers in hires also fell from 39% to 27%.

[2] Arnott [3] constructs a model in which costly commuting is also the main explanation of the spatial mismatch hypothesis.

are heterogeneous in their disutility of transportation (or equivalently in their search costs). These two fundamental assumptions are sufficient to generate an equilibrium in which central city residents experience a higher rate of unemployment than suburban residents and suburban firms create more jobs than central firms (higher job vacancy rate). Their model yields the same policy implications that the two models above since improvements in the efficiency of the matching function and/or in the transportation infrastructure yield a lower level of unemployment. They propose however another policy that is more specific to their model. The government should reduce the differential in the fixed entry cost in order to partially alleviate the spatial mismatch; for example, by subsidizing the entry of firms in the CBD. Such policies have been implemented in the U.S. through the enterprise zone programs (Papke [27], Boarnet and Bogart [6] and Mauer and Ott [25]). The basic idea is to designate a specific urban (or rural) area, which is depressed, and target it for economic development through government-provided subsidies to labor and capital.[3]

In the present paper, we propose an alternative theoretical approach to explain the spatial mismatch hypothesis. Using a search-matching model with endogenous housing consumption and location, we show that distance to jobs is harmful because it implies low search intensities. There is in fact a fundamental trade-off between short-run and long-run benefits of various location choices for the unemployed. Indeed, locations near jobs are costly in the short run (both in terms of high rents and low housing consumption), but allow higher search intensities which in turn increase the long-run prospects of reemployment. Conversely, locations far from jobs are more desirable in the short run (low rents and high housing consumption) but allow only infrequent trips to jobs and hence reduce the long-run prospects of reemployment. Therefore, for workers residing further away from the CBD, it is optimal to spend the minimal search effort whereas workers residing close to jobs provide high search effort.

In this context, spatial mismatch can be the result of optimizing behavior on the part of the labor market participants since the unemployed can *choose* low amounts of search and long-term unemployment. This implies that the standard US-style mismatch arises because inner-city blacks choose to remain in the inner-city and search only little. They do not relocate to the suburbs because the short run-long run gap is big enough to make locations near the jobs

[3] For a general survey on the theoretical foundations of the spatial mismatch, see Gobillon, Selod and Zenou [13].

too expensive. The policy implications are therefore quite different. In particular, "Moving to Opportunity" programs (such as the so-called Gautreaux program) are just the correct policy device to reduce mismatch, rather than lower search costs in some other way.

More precisely, a spatial labor market model is developed in which both job-matching behavior and residential-location behavior are treated simultaneously. Since time is discrete, search intensity is the fraction of the period during which the unemployed are actively searching. Equilibrium for this system involves the interaction of two markets: a spatially concentrated (CBD) labor market in which unemployed workers compete for jobs, and a spatially dispersed land market in which all workers compete for residential land. The most important linkage between these markets is in terms of the differing job-search intensities chosen by unemployed workers at various distances from the CBD.

We first show that there is a non-linear decreasing relationship between the residential distance to jobs of the unemployed and their search intensity s. In fact, individuals living sufficiently close to jobs search every day, $s = 1$, whereas those residing far away provide a minimum search intensity, $s = s_0$. Workers living in between these two areas see a decrease in their search intensity from $s = 1$ to $s = s_0$. We then embed this result (the fact that the unemployed's search intensities are location dependent) into an urban equilibrium in which all individuals (including the unemployed) endogenously choose their residential location. This is one of the main difficulties that we had to overcome. In a classification theorem (see Theorem 2), we show that only three urban configurations are compatible with the decreasing relationship between search intensity and location. These possible equilibrium location patterns are shown to differ only in terms of whether the unemployed workers occupy the central core around the CBD, the periphery of the city, or possibly both.

Finally, since our purpose is to shed some light on the spatial mismatch hypothesis, we focus on two urban equilibria: the *core-periphery urban equilibrium,* in which the unemployed reside either close to jobs (and provide full search intensity $s = 1$) or far away from jobs (and provide a positive minimal level of search $s = s_0$) and the *segregated* equilibrium where the unemployed are always far away from jobs. We show that each equilibrium is unique, and we give a set of sufficient conditions for its existence.

The remainder of the paper is organized as follows. Section 2 sets up the model and describes the land and labor markets. In section 3, we demonstrate our first result, namely the non-linear and decreasing relationship between the residential distance to jobs of the unemployed and their search intensity. Section 4 is devoted to our classification theorem that shows that only three urban configurations are compatible with the negative relationship between search intensity and location. In section 5, we show the existence and the uniqueness of the core-periphery equilibrium and of the segregated equilibrium. Finally, we analyze some of the policy implications of our model in section 6.

2. The model

Consider a population of N *workers* who live in a monocentric city where all jobs are concentrated in the central business district (CBD). All employed workers earn the same prevailing *daily wage*, w, and all unemployed workers receive a *daily unemployment benefit*, b (where it is assumed that $b < w$). Employed workers commute to the CBD each day, and unemployed workers also travel to the CBD to search for jobs. Hence all workers desire to be near the CBD, and compete for residential land on this basis. This urban system is thus characterized by two interdependent markets: a *labor market* in which unemployed workers compete for jobs at the CBD, and a *land market* in which all worker compete for land near to the CBD. We now model each of these markets in turn, and then consider the relevant interactions between them.

2.1. The labor market

Since our focus is on the spatial behavior of workers and their match with firms, we cannot use directly the standard *macroeconomic* matching function (Mortensen and Pissarides [26] and Pissarides [29]). Instead, we need to spell out the micro scenario that leads to a well behaved matching function. For that, the present labor market is based on the model of job-matching behavior developed in Smith and Zenou [38], hereafter referred to as [SZ]. It is in fact a variation of the standard urn-ball model where the system steady state is approximated by an exponential-type matching function as the population becomes large (see among others Hall [15], Pissarides [28], Blanchard and Diamond [5]). Let us describe it more precisely.

In our model it is assumed that (i) jobs are completely specialized in terms of skill requirements, and that (ii) workers are heterogeneous in terms of their skill endowments. Thus *job matching* here constitutes a process whereby heterogeneous workers allocate themselves to jobs with different skill requirements. Heterogeneity of workers does not here imply any superiority or inferiority among their abilities. Rather, all are assumed to possess the same level of general human capital, which is manifested in a variety of different skills (as for example college graduates with degrees in different fields). Hence all workers are assumed to have the same chance of being qualified for any given job, as modeled by a common *qualification probability*, γ.

In this context, the actual *job matching process* can be described as follows. At any point in time (time is discrete), each worker is either employed or unemployed, and only unemployed workers are assumed to search for jobs. Since individual jobs are completely specialized, their creation and closing can be regarded as independent events. In particular, job creations and job closings are here modeled as a simple 'birth and death' process in which 'births' are governed by a *job-creation rate*, λ (denoting the mean number of jobs per worker created each day) and 'deaths' are governed by a *job-closure rate*, ρ (denoting the probability that any currently existing job will be closed on a given day). This process is taken to depend on the general state of economy, and hence is treated as exogenous to the labor market. As mentioned above, the *daily wage*, w, is assumed to be the same for all jobs and (for sake of simplicity) is here assumed to be given exogenously. As in [SZ], the behavioral day-to-day scenario for the job market model on a given day, t, can be summarized as follows:

- At the beginning of day t those unemployed workers currently seeking work travel to the job market (CBD). All current job vacancies are posted, and are offered at the going wage w. Each searcher applies for a single job. No additional prior information about jobs is available, and there is no communication between searchers. Hence searchers choose jobs at random, and more than one searcher may apply for the same job.

- As mentioned above, each job applicant has the same probability, γ, of satisfying all qualifications for the given job. If more than one applicant is qualified for a job, the employer chooses a qualified applicant at random. Otherwise the job is not filled on day t.

- At the end of day t each successful applicant is notified, and is requested to start work on the following day. In addition, decisions are made by employers as to which jobs are no longer profitable and should be closed. For currently active jobs which are closed, layoff notices are distributed to workers. Moreover, for jobs which are filled that day and then closed, the successful (but unlucky) applicants are also given notices. Finally, those currently vacant jobs which are closed are simply removed from the postings at the beginning of the next day. As mentioned above, all jobs (active or vacant) have the same chance, ρ, of being closed on day t.

- In addition, those new job opportunities which have arisen during the day (at rate, λ, per worker) are added to the vacant job postings for the next day.

For the present it is assumed that the residential locations of all workers (both employed and unemployed) are given. In this context, the key decision problem for each unemployed worker is to determine his *search intensity*, s, which we here take to be the fraction of days he travels to the CBD in search of work. If the average value of this fraction over all unemployed workers is designated as the *mean search intensity*, $\overline{s}$, then on any given day, the probability that a randomly sampled worker will appear at the job market is by definition $\overline{s}$. Hence if the unemployment pool is large, then it follows (from the Weak Law of Large Numbers) that the fraction of unemployed workers appearing at the market each day is well approximated by $\overline{s}$. This system parameter, $\overline{s}$, is also assumed to be given for the present.

In this context, it is shown in [SZ] that if jobs creations are characterized by the birth-and-death process described above, then there is a unique steady-state distribution of unemployment and job vacancy levels for each set of parameters $(\rho, \lambda, \overline{s}, \gamma)$. Moreover as population size, N, becomes large, this distribution converges in probability to its mean value, characterized by a steady-state *unemployment rate*, u, (representing the fraction of workers unemployed on each day), and steady-state *vacancy rate*, v, (representing the number of vacant jobs per worker on each day). These steady-state values are given by the unique solution of the following steady-state equations:[4]

$$v + (1 - u) = \lambda / \rho \tag{2.1}$$

$$\rho\ (1 - u) = (1 - \rho)\ u\, \overline{s}\, p_h \tag{2.2}$$

[4] This steady state equilibrium can be compared to that of the standard matching model (Mortensen-Pissarides [26], Pissarides [29]), by noting that the Beveridge curve in their model is very similar to our steady-state condition (2.2). However, we do not use the standard free entry condition to close the labor market equilibrium but our steady-state condition (2.1) results from the underlying birth-death process on vacancies.

where the *hiring probability*, p_h (i.e., the probability that a randomly sampled job searcher will be hired on a given day) is given by:[5]

$$p_h = \frac{v}{u\,\overline{s}}\left[1 - e^{-(\gamma \overline{s} u/v)}\right] \tag{2.3}$$

In particular, it is shown that the limiting number of jobs per worker in the system at steady state is given by λ/ρ. Hence, noting that the number of active jobs per worker is precisely the fraction of employed workers, $1-u$, it follows that equation (2.1) is simply an accounting identity relating the number of vacant jobs and active jobs to total jobs per worker. Similarly, noting that $\rho\,(1-u)$ is the number of active jobs per worker closed on a given day, and that $(1-\rho)\,u\,\overline{s}\,p_h$ is the number of active jobs created on a given day (i.e., the fraction of vacant jobs which are filled and not closed), it follows that equation (2.2) amounts simply to the requirement that the number of active jobs per worker remain constant in the steady state (this equation corresponds to the standard Beveridge curve in the matching literature). If we now let $d = \frac{\lambda}{\rho} - 1$, and solve for v in (2.1) as

$$v = u + d \tag{2.4}$$

then (2.1) through (2.4) are seen to imply that the steady-state unemployment rate, u, must satisfy the single equation

$$\rho\,(1-u) = (1-\rho)\,(u+d)\left(1 - e^{-\gamma\frac{\overline{s}u}{u+d}}\right) \tag{2.5}$$

In terms of our present notation, it is shown in [SZ] (Lemma A.2 and Theorem 1.2) that

Theorem 1 (Labor Market Steady State). *For each mean search intensity, $\overline{s} \in [0,1]$, there exists a unique solution, $u(\overline{s})$, to (2.5). In addition, $u(\overline{s})$ a positive decreasing differentiable function of $\overline{s}$ with $u(0) = 1$.*

Notice in particular that there is always a *positive* unemployment rate in the steady state, regardless of how many jobs are being created. This is

[5] This hiring probability corresponds to the following aggregate matching function:

$$m(u,v) = v\left[1 - e^{-(\gamma s u/v)}\right]$$

which has the standard properties (increasing in both its arguments and concave, and homogeneous of degree 1). Observe that the *individual* probability to find a job for a job seeker with search intensity s is given by:

$$s\,p_h = \frac{s}{\overline{s}}\frac{m(u,v)}{u}$$

a consequence of the *frictional unemployment* inherent in the job-matching process itself. There is always some chance that an unemployed worker will not be hired on a given day, regardless of how many jobs are available.

For our later purposes, it is also important to notice that one can solve for $\overline{s}$ in terms of u in (2.5), and obtain the following explicit form for the inverse function:

$$\overline{s} = \psi(u) = -\left(\frac{u+d}{\gamma u}\right) \ln\left[1 - \left(\frac{\rho}{1-\rho}\right)\left(\frac{1-u}{u+d}\right)\right] \tag{2.6}$$

This relation allows one to determine for each unemployment rate, u, the unique mean search intensity level, $\overline{s}$, which will support u as a steady state.

2.2. The land market

As stated in the introduction, all jobs are assumed to be located at the center (CBD) of a large metropolitan area. In a manner similar to Smith and Zenou [36], this metropolitan area is taken to be representable by a *circular monocentric* city, in which the CBD is the unique center of all business activity and in which all commuting distances are measured as straight-line distances to the CBD. Hence individual *locations*, x, are identified with distances for the CBD. In addition the city is assumed to be *closed* with fixed total population, N.[6] As in the labor market model above (which appealed to large-number approximations), the population, N, is here treated as a continuum in which the influence of individual workers is vanishingly small. Residential land (here synonymous with housing) is rented by workers from absentee landlords. In the terminology of Fujita [12], the present model is thus a *closed city model under absentee land ownership* with *land intensity*,

$$L(x) = 2\pi x \tag{2.7}$$

at each distance x from the CBD. A key point in this model is that individuals are now free to consume any amount of land consistent with their budgets. This relaxation is of particular importance in that it allows unemployed workers to compete for locations near the CBD, by consuming small amounts of land (and living in crowded conditions) if necessary.

As in the labor market model above, workers can in principle change employment states from day to day. Loss of employment involves a change in

[6] This implies in particular that there is no in-migration or out-migration from the city. In addition, there are no births or deaths of workers, so that individuals are assumed to be 'infinitely lived'.

income from the daily wage, w, to the daily unemployment benefit, b, and visa versa. Hence, given the prevailing *rent gradient*, $R(x)$, at each location x, this change of income and employment status may motivate individuals to change their location (or at least in the amount of housing consumed at their current location). All such changes are assumed to be instantaneous, and are governed only by individual utility-maximizing behavior.[7] This decision problem for newly unemployed workers is complicated by the fact that finding a new job will involve some level of search intensity, s. In the labor market setting above, unemployed workers must travel to the CBD to find jobs, so that high levels of search intensity require frequent trips to the CBD. This leads to a fundamental trade-off between short-run and long-run benefits of various location choices for the unemployed. On the one hand, locations near the CBD are costly in the short run (both in terms of high rents and crowded living conditions), but allow higher search intensities which in turn increase the long-run prospects of reemployment. Conversely, locations far from the CBD are more desirable in the short run (low rents and uncrowded conditions) but allow only infrequent trips to the CBD and hence reduce the long-run prospects of reemployment.

To model this basic trade-off, we begin by assuming that all workers have identical preferences among consumptions bundles (q, z) of *land* (*housing*), q, and *composite good*, z, representable by a log-linear utility

$$U(q, z) = q^{\alpha} z^{\beta} \tag{2.8}$$

with $\alpha, \beta > 0$, where it is also assumed that $\alpha + \beta < 1$.[8] However the budget

[7] In particular, there are assumed to be no relocation costs, either in terms of time or money. This is a simplifying assumption, which is quite standard in urban economics. It implies that workers change location as soon as they change employment status. In the context of labor markets in which workers tend to experience long unemployment spells (for example black workers), it is a rather good approximation since, when workers become unemployed, they will be less able to pay land rents and, after some time, they will have to relocate in cheaper places. This assumption could be relaxed by assuming for example that workers only care about their expected utility, i.e. the fraction of their lifetime spent employed and unemployed (this is the case if the discount rate is equal to zero) so that, whatever their employment status, they always stay in the same location. This will however complicate the analysis without changing our main result on the relationship between search intensity and distance to jobs.

[8] This property, which implies 'diminishing marginal utility on rays' [i.e., $U(\lambda q, \lambda z) < \lambda U(q, z)$ for all $\lambda > 0$], insures that the optimal-search-intensity problem discussed below has a differentiable maximum. It is important to emphasize here that (unlike the standard urban economic model) the utility function in (2.8) is necessarily *cardinal* in nature, so that properties such as diminishing marginal utility on rays are behaviorally meaningful. See footnote 9 below for further discussion of this point.

constraints for employed and unemployed workers are different. Each *employed* worker living at location, x, has the standard budget constraint

$$qR(x) + cx + z = w \tag{2.9}$$

where z is taken as the numeraire good with unit price, $R(x)$, is the prevailing (daily) rent per unit of land at x, and where c is the daily round-trip cost of commuting to the CBD. However, an *unemployed* worker at x not only has a different daily income, b, but also has different travel costs depending on his chosen level of search intensity, s. Hence the relevant budget constraint for each such unemployed worker is of the form

$$qR(x) + scx + z = b \tag{2.10}$$

where, for example, searching every other day ($s = 1/2$) would yield an average daily travel cost of $cx/2$. If one denotes the *unemployed state* for workers by '0', and the *employed state* by '1', then maximizing utility (2.8) subject to (2.9) yields the following *land demand for employed workers* at x:

$$q_1(x) = \frac{\alpha}{\alpha+\beta} \cdot \frac{w - cx}{R(x)} \tag{2.11}$$

Similarly, maximizing (2.8) subject to (2.10) yields the following *land demand for unemployed workers* at x:

$$q_0(x) = \frac{\alpha}{\alpha+\beta} \cdot \frac{b - s(x)cx}{R(x)} \tag{2.12}$$

We can now derive the following indirect utility

$$U_1(x) = a(w - cx)^{\alpha+\beta} R(x)^{-\alpha} \tag{2.13}$$

for each *employed* worker at x, where $a = [\alpha/(\alpha+\beta)]^{\alpha}[\beta/(\alpha+\beta)]^{\beta}$ and the following indirect utility

$$U_0(s, x) = a(b - scx)^{\alpha+\beta} R(x)^{-\alpha} \tag{2.14}$$

for each *unemployed* worker at x, where in this case s is now included as a relevant choice variable.[9]

[9] At this point it should be noted that there is a basic difference between the present utility formulation and that in [SZ]. In that paper the basic utility tradeoff for all workers was postulated to be in terms of income versus leisure time. In a spaceless world with no travel costs, it can be argued that *time costs* represent the key variable cost in job search. Such costs of course continue to be important when space is introduced. But in the present model, we have endeavored to keep the framework as simple as possible by focusing only on the *travel costs* associated with spatial job search. A more satisfactory approach would of course encompass both types of costs (including the time spent in travel itself).

3. Optimal search intensities in the city

To model the trade-off outlined above, we focus on the decision problem for an unemployed worker at location x who is currently considering his choice of search intensity, s (which for simplicity can be regarded as the choice of a roulette wheel to use each morning in deciding whether to search that day). To weigh alternative choices, he must evaluate the expected future consumption streams resulting from each choice of s. At each point of time in the future the worker will be in one of two states: unemployed (0) or employed (1). Hence, if we now assume that the present value of future consumption bundles for all workers is representable by a common *utility discount rate*, $\sigma \in (0,1)$, and if we designate the expected discounted utility streams starting in each state as the *lifetime values*, V_0 and V_1, of these states,[10] then (by employing the same arguments as in [SZ]) it can be shown that V_0 and V_1 satisfy the following identities:[11]

$$V_0 = \left(\frac{1-e_0}{1-\sigma}\right) U_0 + e_0\, V_1 \tag{3.1}$$

$$V_1 = \left(\frac{1-e_1}{1-\sigma}\right) U_1 + e_1\, V_0 \tag{3.2}$$

where

$$e_0 = \frac{s\sigma\, p_h}{1-\sigma+s\sigma\, p_h} \tag{3.3}$$

[10] To be more precise, preferences over *consumption streams*, i.e., sequences of daily consumption bundles, $\omega = [(q_t, z_t) : t = 1, 2, ...]$, are taken to be representable by a *discounted utility function* of the form $V(\omega) = \sum_t \sigma^t U(q_t, z_t)$, where U is the utility in (2.8). Behavioral conditions for the existence of such representations (including 'impatience' for consumption and 'time stationarity' of preferences) are given in Koopmans [23]. Of particular importance for our present purposes is uniqueness of these representations: the behavioral discount rate, σ, is *unique*, and the consumption utility, U, is unique up to a linear transformation. Hence utility is necessarily *cardinal* in nature, and in fact, has the same measurement status as money [if it is assumed that $U(0,0) = 0$, as in (2.8)]. It is thus perfectly meaningful to treat $V(\omega)$ as the realization of a well defined random variable, V, with different conditional distributions depending on the initial employment status of the worker. Hence the lifetime values, V_0 and V_1, are the corresponding conditional means of V given initial states '0' and '1', respectively.

[11] It is easy to see that (3.1) and (3.2) correspond to the two following more intuitive Bellman equations:

$$V_0 = U_0 + \sigma\,[s\, p_h V_1 + (1 - s\, p_h) V_0]$$

$$V_1 = U_1 + \sigma\,[\rho V_0 + (1-\rho) V_1]$$

$$e_1 = \frac{\sigma\,\rho}{1-\sigma+\sigma\,\rho} \tag{3.4}$$

(and where dependence of the V's and U's on x and s is suppressed).

By substituting [(3.3),(3.4)] into [(3.1),(3.2)] and solving these equations simultaneously, one may express V_0 and V_1 in terms of U_0 and U_1 as follows:

$$V_0 = \frac{(1-\sigma+\sigma\rho)U_0 + (s\sigma p_h)U_1}{(1-\sigma)(1-\sigma+\sigma\rho+s\sigma p_h)} \tag{3.5}$$

$$V_1 = \frac{(1-\sigma+s\sigma p_h)U_1 + (\sigma\rho)U_0}{(1-\sigma)(1-\sigma+\sigma\rho+s\sigma p_h)} \tag{3.6}$$

Returning to our basic decision problem, suppose that an unemployed worker at x is currently reconsidering his search intensity level, s. To characterize his optimal choice of s as an equilibrium condition, it is convenient to assume that the system is in steady state with some *mean search intensity*, $\overline{s}$. Associated with this mean intensity level is a steady-state *hiring probability* (2.3) which we again denote by $p_h = p_h(\overline{s})$. In addition, we also assume that the current *lifetime values*, V_0 and V_1, of both employed and unemployed workers are constant at all locations (as they must be in equilibrium to ensure that no workers are motivated to relocate). In addition we note that $w > b$ implies desirability of employment, and hence that $V_1 > V_0$ in equilibrium. Under these conditions, we ask whether there is some choice of s for the unemployed worker at x which will improve his current lifetime value, i.e. for which $V_0(s,x) > V_0$. Assuming that perturbations in the search intensity, s, of this single individual cannot influence population values, we may treat both p_h and V_1 as constants in this decision problem. However, U_0 and e_0 are seen from (2.14) and (3.3) to be directly influenced by the choice of s. Hence it follows from these expressions, together with (3.1), that worker's lifetime value, $V_0(s,x)$, can be written as:

$$\begin{aligned} V_0(s,x) &= \left(\frac{1-e_0(s)}{1-\sigma}\right) U_0(s,x) + e_0(s)\;V_1 \\ &= \frac{a(b-scx)^{\alpha+\beta}R(x)^{-\alpha} + \sigma\,p_h\,s\,V_1}{1-\sigma+\sigma\,p_h\,s} \end{aligned} \tag{3.7}$$

Finally, to rule out the possibility of a zero level of optimal search intensity, we assume that some minimal amount of travel to the CBD is required (for purchase of the composite good, z), and hence that there is always some incentive for unemployed workers to live in the city.[12] Assuming that $w > b$ and

[12] If the optimal search intensity for an unemployed worker were zero, then since unem-

that all search costs other than travel are zero, it then follows that *unemployed workers are motivated to apply for jobs on every visit to the CBD.* Hence there is a corresponding *minimal search intensity* level, which we denote by $s_0 > 0$.[13] The relevant decision problem for this unemployed worker is thus to choose a value of $s \in [s_0, 1]$ which maximizes (3.7). Observe also from (2.14) that positive utility is only achievable with positive net income, $b - scx$, so that location choices, x, must always be restricted to the interval $[0, \frac{b}{s_0 c})$. We have the following result:

Proposition 1 (Optimal Search Intensities).

- *At each location x, there is a unique search intensity s that maximizes (3.7).*

- *For any prevailing hiring probability, p_h, and constant lifetime values, V_0, V_1, the optimal search intensity function, $s(x)$, for unemployed workers is given for each location, $x \in [0, \frac{b}{s_0 c})$,by*

$$s(x) = \begin{cases} 1 & \text{for} \quad x \leq x(1) \\ \frac{\alpha+\beta}{1-(\alpha+\beta)} \left[\frac{b}{(\alpha+\beta)cx} - \frac{(1-\sigma)V_0}{\sigma\, p_h (V_1 - V_0)} \right] & \text{for} \quad x(1) < x < x(s_0) \\ s_0 & \text{for} \quad x \geq x(s_0) \end{cases} \tag{3.8}$$

where

$$x(s) = \frac{b}{sc} \cdot \frac{s\sigma\, p_h\ (V_1 - V_0)}{(\alpha+\beta)(1-\sigma)V_0 + [1-(\alpha+\beta)]s\sigma\, p_h\ (V_1 - V_0)} \tag{3.9}$$

Proof. See section A.1 in the Appendix.

The following comments are in order. First, using the first order condition (A.3) in the Appendix, we can easily see the trade off faced by the unemployed when they decide their optimal search intensity level. The left hand side is the short-run utility loss from a marginal increase in search intensity, and the right hand side is the corresponding long-run utility gain from future employment. Indeed, on the one hand, there is a direct and short-run cost of searching more

ployment benefits are taken to be exogenous, there would be no incentive to stay in the city.

[13] We note in passing that the existence of a minimal positive search intensity, s_0, implies that the steady-state mean search intensity, $\overline{s}$, can be no less that s_0, and hence must also be positive.

today $-\partial U_0(s,x)/\partial s$ since it implies higher commuting costs[14] and a lower housing consumption, and thus lower instantaneous utility. On the other, there is a long-run gain of searching more today $\sigma\, p_h\, [V_1 - V_0\,(s,x)]$ since it increases the marginal chance to obtain a job (remember that the individual probability to obtain a job is $s\, p_h$) and the corresponding life-time surplus of being employed. This leads to a fundamental trade-off between short-run and long-run benefits of various location choices for the unemployed. Indeed, locations near the CBD are costly in the short run (both in terms of high rents and low housing consumption), but allow higher search intensities which in turn increase the long-run prospects of reemployment. Conversely, locations far from the CBD are more desirable in the short run (low rents and high housing consumption) but allow only infrequent trips to the CBD and hence reduce the long-run prospects of reemployment. Therefore, for workers residing further away from the CBD ($x \geq x(s_0)$), it is optimal to spend the minimal search effort s_0 whereas it is the contrary ($s = 1$) for workers residing close to jobs ($x \leq x(1)$). Second, this result sheds some light on the spatial mismatch hypothesis. Indeed, as stated in the introduction, distance to jobs is here harmful because it decreases search intensity. Workers who live further away from jobs spend minimal search effort because the short-run gains (low rent and large housing consumption) outweight the long-run gains (higher probability to find a job). Third, from (3.8), it is clear that $s(x)$ is *continuous*, *nonincreasing,* and *strictly decreasing on* $[x(1), x(s_0)]$ (as shown in the top half of Figure 1). Over the decreasing range in particular, this function embodies the continuous trade-off described above. The optimal search intensity $s(x)$ decreases at locations further from the CBD, as unemployed workers compensate for losses in long-run job prospects by short-run gains in net income (maintaining a constant lifetime value level, V_0). Finally, if we take the value of $s(x)$ for interior locations, i.e. for $x(1) < x < x(s_0)$, it is easy to verify that it varies negatively with commuting costs c and the lifetime value of the unemployed V_0, and positively with the hiring probability p_h and the lifetime value of the employed V_1. The intuition is straightforward since when c or V_0 is high and when p_h or V_1 is low, then workers reduce their search effort since either costs of searching are too high or the rewards of searching are too low. Concerning the discount rate σ, one can verify that it is positively correlated with $s(x)$ so that putting more weight on today's gain increases search intensity.

[14] Commuting costs have to be taken here in a broader sense as long as it measures access to employment activities. For example, including time commuting costs in our framework will imply that the marginal cost of an increased search leads to a reduced leisure time.

Is this result consistent with empirical studies? In fact, most studies have shown that workers' search intensity is negatively related to their residential distance to jobs. For example, Seater [34] has found that workers searching further away from the residence are less productive than those who search closer to where they live. Barron and Gilley [7] and Chirinko [10] have also found that there are diminishing returns to search when people live far away from jobs. Rogers [31] has also demonstrated that access to employment is a significant variable in explaining the probability of leaving unemployment.

[*Insert Figure* 1 *here*]

4. The different urban land use equilibria

So far, we have determined the optimal search intensity of the unemployed at each location in the city. The key question now is how the urban land use equilibrium looks. In other words, knowing this function $s(x)$, where do the unemployed and the employed locate in the city? The basic trade-off for the employed is between commuting costs and housing consumption whereas for the unemployed, it is between commuting/search costs, housing consumption and search intensity (and thus the duration of unemployment). In order to determine the urban land use equilibrium, we have to define the bid rent function of each group of workers.[15]

4.1. Bid rents and locational equilibrium patterns

Given the utilities and lifetime values above, we now define the equilibrium bid-rents which are possible for any set of equilibrium values (p_h, V_0, V_1) with $V_1 > V_0$. Turning first to employed workers, we may observe from (3.2) and (3.4) that their equilibrium utility level, U_1, is constant over locations, and is given by

$$\begin{aligned} U_1 &= \left(\frac{1-\sigma}{1-e_1}\right)(V_1 - e_1 V_0) \\ &= (1-\sigma)V_1 + \sigma\rho(V_1 - V_0) \end{aligned} \tag{4.1}$$

Hence it follows from the form of the indirect utility in (2.13) that the relevant *bid rent function*, $R_1(x)$, for employed workers at each location, $x \in [0, \frac{w}{c})$, is

[15] The bid rent is a standard concept in urban economics. It indicates the maximum land rent that a worker located at a distance x from the CBD is ready to pay in order to achieve the equilibrium utility level of his/her group.

given by the relation:

$$a(w-cx)^{\alpha+\beta}R_1(x)^{-\alpha} = U_1 = (1-\sigma)V_1 + \sigma\rho(V_1 - V_0)$$

$$\Rightarrow R_1(x) = \left[\frac{a(w-cx)^{\alpha+\beta}}{(1-\sigma)V_1 + \sigma\rho(V_1-V_0)}\right]^{\frac{1}{\alpha}} \tag{4.2}$$

The bid rent function for unemployed workers is considerably more complex, in that it depends on the optimal search intensity level at each location. To specify this function observe first from (3.1) and (3.3) that the equilibrium utility, $U_0(x)$, at each location, $x \in [0, \frac{b}{s_0 c})$ is given [in a manner paralleling (4.1)] by

$$\begin{aligned} U_0(x) &= \left(\frac{1-\sigma}{1-e_0(x)}\right)(V_0 - e_0(x)V_1) \\ &= (1-\sigma)V_0 - s(x)\sigma p_h(V_1 - V_0) \end{aligned} \tag{4.3}$$

Hence the indirect utility in (2.14) yields the following *bid rent function*, $R_0(x)$, for unemployed workers at each location, $x \in [0, \frac{b}{s_0 c})$:

$$a[b - s(x)cx]^{\alpha+\beta}R_0(x)^{-\alpha} = U_0(x) = (1-\sigma)V_0 - s(x)\sigma p_h(V_1 - V_0)$$

$$\Rightarrow R_0(x) = \left[\frac{a[b-s(x)cx]^{\alpha+\beta}}{(1-\sigma)V_0 - s(x)\sigma p_h(V_1-V_0)}\right]^{\frac{1}{\alpha}} \tag{4.4}$$

where $s(x)$ is given by (3.8) above. [An instance of this (piecewise continuously differentiable) bid rent function is shown in the bottom half of Figure 1, where the curve represents a typical 'slice' through the two-dimensional rent surface].

It should be clear that the bid rents are calculated such that the lifetime utilities of both the employed and the unemployed workers, respectively, V_1 and V_0, are spatially invariant. Compare for example an unemployed worker residing close to jobs and another unemployed worker living far away from jobs. The former has a lower search (commuting) cost and a higher chance to find a job but consume less land whereas the latter has a higher search (commuting) cost and a lower chance to find a job but consume more land. The bid rent defined by (4.4) exactly compensates these differences by ensuring that these two workers obtain the same lifetime utility V_0. This is not true for the current utility of the unemployed $U_0(x)$ because, as can be seen in (4.3), the land rent does not compensate for $s(x)$. In fact, the unemployed residing close to jobs have a lower current utility than the ones living far away from jobs because they provide more search intensity (indeed, using (4.3), it is easy to see that $U_0'(x) > 0$). However, because they provide more search intensity, they have a higher chance to find a job, and thus in the long-run they compensate the short-run disadvantage so that all unemployed workers obtain V_0.

If there is also postulated to be an exogenous level of *agricultural rent* (or *opportunity rent*), R_A, which is uniform in space, then it follows by standard competitive arguments land at each location is assigned to the highest bidder. This implies in particular that the *equilibrium land rent function*, $R(x)$, must be given at all locations, x, by

$$R(x) = \max\{R_0(x), R_1(x), R_A\} \tag{4.5}$$

In addition, land at x can only be occupied by workers (employed or unemployed) if their bid rents are maximal. More precisely, if the *population densities* of employed workers and unemployed workers at x are denoted respectively by $\eta_1(x)$ and $\eta_0(x)$, then at equilibrium we must have

$$\eta_i(x) > 0 \Rightarrow R_i(x) = R(x)\ ,\ i = 0, 1 \tag{4.6}$$

Finally, we have the usual 'land capacity' condition that no more land be consumed than is available, and 'land filling' condition that all land with rents higher than agricultural rent must be occupied by workers. To state these conditions precisely, observe that, from above, the optimal land demand for employed workers at x is given by (2.11) whereas the optimal land demand for unemployed workers at x is given by (2.12), [with $s = s(x)$]. In terms of these land demands, the *land capacity condition* and *land filling condition* take the respective forms [see for example in Fujita (1989, p.102)]

$$q_0(x)\eta_0(x) + q_1(x)\eta_1(x) \leq L(x) \tag{4.7}$$

$$R(x) > R_A \Rightarrow q_0(x)\eta_0(x) + q_1(x)\eta_1(x) = L(x) \tag{4.8}$$

where $L(x) = 2\pi x$. Conditions [(4.6),(4.7)(4.8)] can be given a sharper form in the present model as we now show.

4.2. Classification of equilibrium land use patterns

With the non-linear bid rents defined by (4.2) and (4.4), different urban configurations can emerge. Indeed, the land market being perfectly competitive, all workers propose different bid rents at different locations and (absentee) landlords allocate land to the highest bids. So depending on the different steepness of the bid rents (as captured by their slopes), at each location, the employed can outbid the unemployed or can be outbidden by the unemployed. An example of the equilibrium rent function defined by (4.5) is shown in Figure 2.

In particular, this figure illustrates a case where unemployed workers occupy both a central core of locations and a peripheral ring of locations about the CBD, separated by an intermediate ring of employed workers. Other urban configurations may also emerge. For example, the unemployed can occupy the core of the city and the employed the suburbs. The reverse pattern may also prevail. Since we want to focus on interesting urban configurations in which the unemployed workers can outbid the employed workers for peripheral land in equilibrium, we assume

$$w < \frac{b}{s_0} \tag{4.9}$$

Because this possibility is of considerable interest for our present purposes, we shall assume (4.9) throughout the analysis to follow.[16]

But while (4.9) does allow for this possibility, it is by no means sufficient. Hence the main result of this section is to show that the conditions above imply that in equilibrium there are exactly *three* possible locational configurations of workers:

Theorem 2 (Equilibrium Location Patterns). *In equilibrium there are exactly three possible locational patterns:*
(*i*) *a central core of unemployed surrounded by a peripheral ring of employed,*
(*ii*) *a central core of employed surrounded by a peripheral ring of unemployed,*
(*iii*) *both a central core and peripheral ring of unemployed separated by an intermediate ring of employed.*

Proof. See section A.2 in the Appendix.

This theorem shows that, in a framework where workers' search intensity is location dependent (see Proposition 1), different urban equilibrium configurations can emerge. In the first one (*i*), referred to as the *Integrated Equilibrium*, the unemployed reside close to the CBD, have high search intensities and experience short unemployment spells. In the second one (*ii*), referred to as the *Segregated Equilibrium*, the employed occupy the core of the city and bid away the unemployed in the suburbs. In this case, the latter tend to stay unemployed for a longer time since their search intensity is quite low. Finally, the third case (*iii*), referred to as the *Core-Periphery Equilibrium*, is when there

[16] Observe that if one relaxes condition (4.9) and instead assumes $w > b/s_0$, then it strongly restricts the set of urban equilibria since the equilibrium that is more likely to prevail is the one where the unemployed reside close to jobs and the employed at the periphery of the city. If condition (4.9) holds, then the analysis is much more richer since three types of urban equilibria can emerge, including the one described above.

are two categories of unemployed: the short-run ones who reside close to jobs and the long-term ones who live at the periphery of the city (see Figure 2).

In Wasmer and Zenou [41] where the relationship between search effort and distance to jobs is assumed instead of being derived (like here), only two equilibria can emerge: (i) and (ii). We would thus like now to study the third type of equilibrium, the core-periphery equilibrium, since it has not yet been investigated, even though it is quite relevant. Furthermore, this equilibrium encompasses the two other ones since the first equilibrium (i) is a limiting case of the core-periphery equilibrium when $x_p = x_f$ (x_p is the border between the employed and the long term-unemployed workers, and x_f is the city-fringe; see Figure 2) while the second equilibrium (ii) is a limiting case of the core-periphery equilibrium when $x_c = 0$ (x_c is the border between the short-run unemployed and the employed workers; see Figure 2).

The key question is to see under which conditions what equilibrium prevails. Since we know from (4.2) and (4.4) that both bid rents $R_1(x)$ and $R_0(0)$ are continuous, twice differentiable, decreasing and convex, we have:

(1) If $R_1(0) > R_0(0)$ and $R_1(x_f) < R_0(x_f)$, then there is a unique Segregated Equilibrium (ii);

(2) If $R_1(0) < R_0(0)$ and $R_1(x_f) > R_0(x_f)$, then there is a unique Integrated Equilibrium (i);

(3) If $R_1(0) < R_0(0)$, $R_1'(0) < R_0'(0)$ and $R_1(x_f) < R_0(x_f)$, then there is a unique Core-Periphery Equilibrium (iii).

Of course, because it is so cumbersome (since x_f, V_0, V_1, p_h and ρ are all endogenous variables), the exact conditions on the exogenous parameters are impossible to determine analytically. We have here implicit conditions that link endogenous and exogenous variables.

Let us now focus on the more general equilibrium (iii) (since the others are just a particular case of (iii)). We will characterize it, shows its existence and uniqueness.

$[Insert\ Figure\ 2]$

5. The Core-Periphery equilibrium

To define the core-periphery equilibrium, we first collect all the relevant parameters for the problem. For any probabilities, $\rho, \sigma, \gamma, s_0 \in (0, 1)$, and scalars, $\lambda, \alpha, \beta, b, w, c, N, R_A > 0$ with $\alpha + \beta < 1$ and $s_0 w < b < w$, we may define an *admissible parameter vector*, $\theta = (\rho, \sigma, \gamma, s_0, \lambda, N, \alpha, \beta, b, w, c, R_A)$. Next, for any given *lifetime values*, V_0, V_1, with $V_1 > V_0$, and *hiring probability*, $p_h \in (0,1)$, we define the following set of functions. First, let the function s be defined by (3.8) with ranges, $x(1)$ and $x(s_0)$, given by (3.9). In terms of s and (V_0, V_1, p_h), we may then define the additional functions, U_0, R_0, R_1,and R, respectively by (4.3), (4.4), (4.2), (4.5). Using R_0, R_1, and R, we next define the *indicator functions*, $\delta_i, i = 0, 1$, specifying the relevant regions occupied by unemployed and employed workers, respectively:

$$\delta_i(x) = \begin{cases} 1 & , \ R_i(x) = R(x) \\ 0 & , \ otherwise \end{cases} \quad , \quad i = 0, 1 \tag{5.1}$$

It should be noted that the validity of this characterization of the location pattern is made possible by the more technical version of Theorem 2 proved in the Appendix (Theorem A.1 plus Lemma 5) which shows that these indicator functions are ambiguous only on a set of measure zero [i.e., that the equality $R_0(x) = R_1(x)$ holds only on a set of measure zero in the interval of relevant distances, x]. Hence one can now sharpen the general set of locational equilibrium conditions [(4.6),(4.7),(4.8)] above by noting in the present case that at almost every distance, x, at most one of the population densities, $\eta_0(x)$ and $\eta_1(x)$, can be positive. Hence, by substituting (2.11) and (2.12) into (4.8), and observing that by definition, $R_i(x) = R(x)$ iff $\delta_i(x) = 1$, it follows that the appropriate *population densities*, must have the form

$$\eta_0(x) = \frac{L(x)}{q_0(x)} = 2\pi x \left(\frac{\alpha+\beta}{\alpha}\right) \frac{R_0(x)}{b - s(x)cx} \tag{5.2}$$

$$\eta_1(x) = \frac{L(x)}{q_1(x)} = 2\pi x \left(\frac{\alpha+\beta}{\alpha}\right) \frac{R_1(x)}{w - cx} \tag{5.3}$$

At this point, it is important to reiterate that all of the above functions are completely defined by the *lifetime values*, V_0, V_1, and *hiring probability*, p_h. With these functions, we can now give a formal general definition of equilibrium as follows:

Definition 1 (General). *For any admissible parameter values,* $\theta = (\rho, \gamma, s_0, \lambda, N, \sigma, \alpha, \beta, b, w, c, R_A)$, *a nonnegative vector* $\xi = (V_0, V_1, p_h, u, \overline{s}, N_0, N_1)$ *is said to be an* equilibrium *for* θ *iff* ξ *satisfies the following five conditions [where* $d = 1 - \frac{\lambda}{\rho}$*, and where the functions* $(s, U_0, R_0, R_1, R, \delta_0, \delta_1, \eta_0, \eta_1)$ *are given by the constructions above]:*

$$p_h = \frac{u+d}{u\overline{s}} \left(1 - e^{-\gamma \frac{u\overline{s}}{u+d}}\right) \tag{5.4}$$

$$\rho(1-u) = (1-\rho)u\overline{s}p_h \tag{5.5}$$

$$\overline{s} = \frac{1}{N_0}\int s(x)\delta_0(x)\eta_0(x)dx \tag{5.6}$$

$$N_i = \int \delta_i(x)\eta_i(x)dx\,,\ \ i = 0,1 \tag{5.7}$$

$$N = N_0 + N_1 \tag{5.8}$$

The first two conditions follow from [(2.3),(2.4),(2.5)] and define the labor market steady state, given the mean search intensity, $\overline{s}$. Condition (5.6) defines $\overline{s}$ in terms of the search intensities, $s(x)$, and population densities, $\eta_0(x)$, at each location x occupied by unemployed workers [i.e., with $\delta_0(x) = 1$]. Finally, condition (5.7) defines the population totals for employed and unemployed workers, together with the accounting condition (5.8) that all workers are either employed or unemployed.[17]

While this definition is conceptually quite simple in that it gives a *finite-dimensional* characterization of equilibrium [in terms of the scalar variables $(V_0, V_1,\ p_h, u, \overline{s}, N_0, N_1)$], it is not very tractable analytically. In particular, indicator functions such as δ_0 and δ_1 are difficult to analyze in practice. However, by employing Theorem 2 (and its more technical counterpart, Theorem A.1 in the Appendix), one can give a more explicit characterization of these indicator functions. In particular, it follows from Theorem 2 that employed workers will always live in a single connected ring, and hence that the positive support of the indicator function, δ_1, must be closed interval, $[x_c, x_p]$, with end points given by[18]

$$x_c = \min\{x \geq 0 : \delta_1(x) > 0\} \tag{5.9}$$

$$x_p = \max\{x \geq 0 : \delta_1(x) > 0\} \tag{5.10}$$

In addition, it follows that unemployed workers will live in at most two distinct rings, the first given by $[0, x_c]$ and the second by $[x_p, x_f]$, where x_f is the *frontier location* (or *city edge*) as characterized by

$$x_f = \min\{x \geq 0 : R(x) = R_A\} \tag{5.11}$$

Hence, in the present case, it is possible to remove the indicator functions above, and replace [(5.6),(5.7)] by a more explicit set of conditions involving only the density functions (η_0, η_1) and the boundary variables (x_c, x_p, x_f).

[17] Note that the bid-rent and population density conditions [(4.5),(4.6),(4.7),(4.8)] stated above are not made explicit in this formulation, but rather are implicit in the definitions of the indicator functions, δ_0 and δ_1.

[18] Given the possibility of 'trivial intersection points' (as in Lemma 5 of the appendix), a more techically correct version of these conditions would be to replace 'min' in (5.9) by 'essential infimum' and 'max' in (5.10) by 'essential supremum'.

This plan is now carried out for an important subclass of equilibria (the core-periphery ones), which illustrate all the main features of the above model, and which are sufficiently tractable to allow a detailed analysis of equilibria. The equilibrium bid-rent configuration shown in Figure 2 yields a simple type of core-periphery location pattern. Notice in particular that there are only *two* search intensity levels for unemployed workers: all unemployed workers in the central core search with *full intensity*, $s = 1$, and all in the peripheral ring search with *minimum intensity*, $s = s_0$. These constant-search-intensity patterns are particularly easy to analyze, as should be evident from (3.8). Moreover, Theorem 2 shows that essentially all equilibrium properties of the system can be studied in terms of these simple cases. For in the other two possible locational patterns, it is clear that so long as the equilibrium bid-rent curves, R_0 and R_1, do not cross in the region $[x(1), x(s_0)]$, only maximal and minimal search intensities will be involved. Moreover, the case illustrated in Figure 2, where the region $[x(1), x(s_0)]$ is shown to be relatively small, is in fact quite typical. This assertion is supported by the following result, which shows that if utility is 'almost linearly homogeneous' in the sense that $\alpha + \beta$ is close to one, then the interval $[x(1), x(s_0)]$ is necessarily very small:

Proposition 2. *If* $\alpha + \beta \approx 1$, *then in equilibrium* $|x(1) - x(s_0)| \approx 0$.

Proof: It is enough to observe from (3.9) that for any given lifetime values and hiring probability (V_0, V_1, p_h), the locations $x(1)$ and $x(s_0)$ have a common limiting value, $\frac{b}{c}\frac{\sigma p_h(V_i - V_0)}{(1-\sigma)V_0}$, as $\alpha + \beta \to 1$. ■

Hence if diminishing marginal utility (along rays) is sufficiently small, then equilibrium can be safely assumed to involve only maximal and/or minimal search intensities for unemployed workers.

With these observations, we now restrict attention to the constant-search-intensity case. In particular, we focus on the class of *core-periphery equilibria*, which involve both constant maximal search intensity in a central unemployment core $[0, x_c]$, and minimal search intensity in a peripheral unemployment ring $[x_p, x_f]$. The other two equilibrium possibilities (equilibria (i) and (ii)) with constant search intensities can then be regarded as limiting cases in which either $x_c = 0$ or $x_p = x_f$.

Our first objective is to give a formal definition of core-periphery equilibria which specializes the general definition above, and which allows a more detailed analysis of both existence and uniqueness properties. To do so, we first observe from (4.3) that in equilibrium, $U_0(x) \equiv U_0[s(x)]$, so that each region with constant search intensity must necessarily involve constant utility. For unemployed workers in the core region (with $s = 1$), this equilibrium *core utility level*, U_0, must satisfy

$$U_0^c = (1 - \sigma)V_0 - \sigma p_h(V_1 - V_0) \tag{5.12}$$

and for those in the peripheral region (with $s = s_0$), the corresponding *peripheral utility level*, which we denote by U_0^p, must satisfy

$$U_0^p = (1 - \sigma)V_0 - s_0\sigma p_h(V_1 - V_0) \tag{5.13}$$

Moreover, by evaluating (3.5) at both $s = 1$ and $s = s_0$, we obtain the identity

$$\frac{(1 - \sigma + \sigma\rho)U_0^c + (\sigma p_h)U_1}{1 - \sigma + \sigma\rho + \sigma p_h} = \frac{(1 - \sigma + \sigma\rho)U_0^p + (s_0\sigma p_h)U_1}{1 - \sigma + \sigma\rho + s_0\sigma p_h} \tag{5.14}$$

which can be solved for U_0^p to yield

$$U_0^p = \tau(p_h)U_0^c + [1 - \tau(p_h)]U_1 \tag{5.15}$$

where

$$\tau(p_h) = \frac{(1 - \sigma + \sigma\rho) + s_0\sigma p_h}{(1 - \sigma + \sigma\rho) + \sigma p_h} \in (0, 1) \tag{5.16}$$

It is worth noting at this point that since $w > b$ of course implies that $U_1 > U_0^c$ in equilibrium, and since the positivity of steady-state unemployment levels, u (Theorem 1) implies from (5.4) that steady-state hiring probabilities, p_h, are always positive, it follows from the convex combination in (5.15) that in every core-periphery equilibrium one must have

$$U_0^c < U_0^p < U_1 \tag{5.17}$$

This again underscores the essential difference between unemployed workers in the central core and those in the periphery. Those in the central core are giving up short-run utility for long-run utility gains. Hence, if the lifetime value, V_0, of all unemployed workers is the same, then the short-run utility of those in the periphery must be greater than for those in the central core. These constant utility levels (U_0^c, U_0^p, U_1) will also turn out to be more useful for analysis than the more general lifetime values (V_0, V_1). Hence the present equilibrium conditions will be developed in terms of (U_0^c, U_0^p, U_1).

Next we observe that the (outer) *core boundary point*, x_c, and the (inner) *peripheral boundary point*, x_p, can now be characterized as intersections between these constant-utility curves as follows. First observe that since the bid rent for core unemployed workers and employed workers must be the same at x_c, it follows from (2.13) and(2.14) that in equilibrium,

$$\frac{U_0^c}{U_1} = \left(\frac{b - cx_c}{w - cx_c}\right)^{\alpha+\beta} \tag{5.18}$$

Similarly, since the bid rent for peripheral unemployed workers and employed workers must be the same at x_p, it also follows from (2.13) and (2.14) that in equilibrium,

$$\frac{U_0^p}{U_1} = \left(\frac{b - s_0 cx_p}{w - cx_p}\right)^{\alpha+\beta} \tag{5.19}$$

A final consequence of these constant utility levels is to yield more explicit expressions for the population densities in (5.2) and (5.3). First, by solving for rent $R(x)$ in (2.13) and substituting this into (2.11) it follows from (5.3) that the equilibrium *employment density*, $\eta_1(x)$, is now given for all $x \in [x_c, x_p]$ by

$$\eta_1(x) = 2\pi x \left(\frac{\alpha+\beta}{\alpha}\right)\left(\frac{a}{U_1}\right)^{\frac{1}{\alpha}} (w - cx)^{\frac{\beta}{\alpha}} \tag{5.20}$$

Similarly, by setting $s(x) = 1$, solving for $R(x)$ in (2.14), and substituting this into (2.12), it follows from (5.2) that the equilibrium *core unemployment density*, $\eta_0^c(x)$, is given for all $x \in [0, x_c]$ by

$$\eta_0^c(x) = 2\pi x \left(\frac{\alpha+\beta}{\alpha}\right)\left(\frac{a}{U_0^c}\right)^{\frac{1}{\alpha}} (b - cx)^{\frac{\beta}{\alpha}} \tag{5.21}$$

The same procedure with $s(x) = s_0$ also yields the equilibrium *peripheral unemployment density*, $\eta_0^p(x)$, defined for all $x \in [x_p, x_f]$ by

$$\eta_0^p(x) = 2\pi x \left(\frac{\alpha+\beta}{\alpha}\right)\left(\frac{a}{U_0^p}\right)^{\frac{1}{\alpha}} (b - s_0 cx)^{\frac{\beta}{\alpha}} \tag{5.22}$$

Given these population densities and corresponding boundary points, it follows that the integrals in (5.7) can now be calculated explicitly. In particular, if N_1 again denotes the equilibrium *employment level*, and if N_0^c and N_0^p now denote the equilibrium *core unemployment level* and *peripheral unemployment level*, respectively, then N_0^c can be calculated explicitly as

$$\begin{aligned} N_0^c &= \int_0^{x_c} \eta_0^c(x)dx \\ &= 2\pi\left(\frac{\alpha+\beta}{\alpha}\right)\left(\frac{a}{U_0^c}\right)^{\frac{1}{\alpha}} \left\{-\left(\frac{\alpha x_c}{c(\alpha+\beta)}\right)(b - cx_c)^{\frac{\alpha+\beta}{\alpha}} + \right. \\ &\left. \left(\frac{\alpha}{c(\alpha+\beta)}\right)\left(\frac{\alpha}{c(2\alpha+\beta)}\right)\left[b^{\frac{2\alpha+\beta}{\alpha}} - (b - cx_c)^{\frac{2\alpha+\beta}{\alpha}}\right]\right\} \end{aligned} \tag{5.23}$$

Similarly, N_1 is now given by:

$$\begin{aligned} N_1 &= \int_{x_c}^{x_p} \eta_1(x)dx \\ &= 2\pi\left(\frac{\alpha+\beta}{\alpha}\right)\left(\frac{a}{U_1}\right)^{\frac{1}{\alpha}}\left\{\left(\frac{\alpha x_c}{c\,(\alpha+\beta)}\right)(w-cx_c)^{\frac{\alpha+\beta}{\alpha}} - \right. \\ & \left(\frac{\alpha x_p}{c\,(\alpha+\beta)}\right)(w-cx_p)^{\frac{\alpha+\beta}{\alpha}} + \left(\frac{\alpha}{c\,(\alpha+\beta)}\right)\left(\frac{\alpha}{c\,(2\alpha+\beta)}\right)\cdot \\ & \left.\left[(w-cx_c)^{\frac{2\alpha+\beta}{\alpha}} - (w-cx_p)^{\frac{2\alpha+\beta}{\alpha}}\right]\right\} \end{aligned} \tag{5.24}$$

and N_0^p is given by:

$$\begin{aligned} N_0^p &= \int_{x_p}^{x_f} \eta_0^p(x)dx \\ &= 2\pi\left(\frac{\alpha+\beta}{\alpha}\right)\left(\frac{a}{U_0^p}\right)^{\frac{1}{\alpha}}\left\{\left(\frac{\alpha x_p}{s_0c\,(\alpha+\beta)}\right)(b-s_0cx_p)^{\frac{\alpha+\beta}{\alpha}} - \right. \\ & \left(\frac{\alpha x_f}{s_0c\,(\alpha+\beta)}\right)(b-s_0cx_f)^{\frac{\alpha+\beta}{\alpha}} + \left(\frac{\alpha}{s_0c\,(\alpha+\beta)}\right)\left(\frac{\alpha}{s_0c\,(2\alpha+\beta)}\right)\cdot \\ & \left.\left[(b-s_0cx_p)^{\frac{2\alpha+\beta}{\alpha}} - (b-s_0cx_f)^{\frac{2\alpha+\beta}{\alpha}}\right]\right\} \end{aligned} \tag{5.25}$$

Given these equilibrium population levels, we next observe that the single most important simplification made possible by present constant-search-intensity hypothesis is the determination of the equilibrium *mean search intensity level*, $\overline{s}$. In particular, since the relevant search intensity function $s(x)$ has only two values, it now follows that equilibrium condition (5.6) can be replaced by the much simpler form

$$\overline{s} = \frac{N_0^c + s_0 N_0^p}{N_0^c + N_0^p} \tag{5.26}$$

Hence the steady-state model of the labor market can be completely specified in terms of the three population variables (N_0^c, N_0^p, N_1). In particular, since the equilibrium *unemployment rate*, u, is given by

$$u = \frac{N_0^c + N_0^p}{N} = \frac{N - N_1}{N} \tag{5.27}$$

it follows that the inverse relation in (2.6) now yields a single equilibrium condition relating N_0^c and N_0^p:

$$\frac{N_0^c + s_0 N_0^p}{N_0^c + N_0^p} = \psi\left(\frac{N - N_1}{N}\right) \tag{5.28}$$

This, together with the accounting identity

$$N_0^c + N_0^p + N_1 = N \tag{5.29}$$

allows one to determine unique values of N_0^c and N_0^p for each employment level, N_1. In addition, by substituting (5.26) and (5.27) into (5.4) it follows that the hiring probability p_h can then be determined as

$$p_h = \frac{N_0^c + s_0 N_0^p}{N_0^c + N_0^p + Nd}\left(1 - e^{-\gamma \frac{N_0^c + s_0 N_0^p}{N_0^c + N_0^p + Nd}}\right) \tag{5.30}$$

To complete the equilibrium conditions for the present core-periphery case, recall that the boundary points, (x_c, x_p, x_f) must satisfy certain additional consistency conditions. First, it follows by hypothesis that full search intensity, $s = 1$, is optimal for core unemployed workers, and hence from (3.8) that the core boundary point, x_c, must satisfy

$$x_c \leq x(1) \tag{5.31}$$

Similarly, minimal search intensity, $s = s_0$, is assumed to be optimal for peripheral unemployed workers, so that the peripheral boundary point, x_p, must satisfy

$$x_p \geq x(s_0) \tag{5.32}$$

Finally, it also follows by definition that bid rent for peripheral unemployed workers must equal the agricultural rent, R_A, at the frontier location, x_f. Hence, by letting $x = x_f$, $s(x) = s_0$, and $U_0(x) = U_0^p$ in (2.14) [or (4.4)] it follows that at the frontier location we must have

$$R_A = \left(\frac{a}{U_0^p}\right)^{\frac{1}{\alpha}} (b - s_0 c x_f)^{\frac{\alpha+\beta}{\alpha}} \tag{5.33}$$

This completes the set of equilibrium conditions for the core-periphery case. Hence we have:

Definition 2 (*CP*-Equilibrium). *For any admissible parameter vector,* $\theta = (\rho, \gamma, s_0, \lambda, N, \sigma, \alpha, \beta, b, w, c, R_A)$, *a vector of values,* $\xi = (N_0^c, N_0^p, N_1, p_h, U_0^c, U_0^p, U_1, x_c, x_p, x_f)$, *is said to be* a core-periphery (CP) equilibrium for θ *iff conditions [(5.15),(5.18),(5.19),(5.23), (5.24),(5.25),(5.28),(5.29),(5.30),(5.31),(5.32),(5.33)] are satisfied.*

Given this definition, we are now able to show the existence and the uniqueness of the core-periphery (CP) equilibrium. The proofs of the existence and uniqueness of the core-periphery equilibrium are quite complex since they involve two markets that are totally integrated. We have first the following uniqueness property of core-periphery equilibria (proved in section A.3 in the Appendix):

Theorem 5.1 (Uniqueness of *CP*-Equilibria). *For each vector of admissible parameters,* $\theta = (\rho, \gamma, s_0, \lambda, N, \sigma, \alpha, \beta, b, w, c, R_A)$*, there is at most one CP-equilibrium for* θ*.*

Furthermore, the existence of a core-periphery equilibrium is established by the following result (proved in section A.3 in the Appendix):

Theorem 5.2 (Existence of *CP*-Equilibria). *For any admissible parameters,* $\widehat{\theta} = (\rho, \gamma, s_0, \lambda, N, \sigma, \alpha, \beta, w, c)$*, and any* $b \in (\widehat{b}, w)$ *in Proposition 7 with corresponding minimal and maximal equilibria,* $\overline{\xi}(\widehat{\theta}, b)$*,* $\overline{\overline{\xi}}(\widehat{\theta}, b)$*, there exists for each agricultural rent level,* R_A*, in the interval* (A.171) *a unique CP-equilibrium for* $\theta = (\widehat{\theta}, b, R_A)$*.*

6. Discussions and policy implications

In our model, there is room for policies because, as in the standard search-matching literature (Mortensen and Pissarides [26], Pissarides [28]), market failures are caused by search externalities. There are in fact two types of search externalities: *negative intra-group externalities* (more searching workers reduces the job-acquisition rate) and *positive inter-group externalities* (more searching firms increases the job-acquisition rate).

We would like now to show how the present paper provides a new economics mechanism of the spatial mismatch hypothesis and thus new policy implications. Since our goal is to give a theoretical explanation of the spatial mismatch hypothesis, we focus now on equilibria (*ii*) (the *Segregated Equilibrium*) and (*iii*) (the *Core-Periphery Equilibrium*) because in both cases some (or all) unemployed reside far away from jobs.

In the segregated equilibrium, the unemployed *decide* to reside far away from jobs and thus *voluntary choose* low amounts of search and long-term unemployment. In this context, the standard US-style mismatch arises because inner-city blacks choose to remain in the inner-city and search only little. They do not relocate to the suburbs (in our model this is the core, but in the US mismatch it is the suburbs) because the short- run gains (low rent and large housing consumption) outweigh the long-run gains of residing near jobs (higher probability of finding a job). *In the segregated equilibrium, the spatial mismatch stems from voluntary choices of workers and not from imposed restricted mobility such as housing discrimination.*

In the core-periphery equilibrium (Figure 2), the unemployed are indifferent between residential locations close and distant to jobs. However, if for some reason like e.g. housing discrimination (see Yinger [42], for empirical evidence), blacks are forced to live further from jobs, then this equilibrium provides another rational explanation for the spatial mismatch hypothesis. Living far away from jobs is harmful, not because of bad information about jobs, but because of too low search effort from workers. In other words, living in areas distant to jobs and in which housing rents are low does not induce the unemployed to put a lot of effort in their search. They are happy to live on welfare since it covers their housing costs and provide enough instantaneous utility. Thus, *in the core-periphery equilibrium, the spatial mismatch stems from involuntary choices of workers, such as for example housing discrimination.*

Even though the causes of spatial mismatch are different in these two equilibria, the consequences are similar. In both cases, the unemployed workers provide too little search effort and thus tend to have long unemployment spells because they prefer short-run over long-run gains. In other words, *the opportunity costs (captured here by land rents, density and leisure time or commuting costs) of not working (or even not participating to the labor market) are too low to motivate these workers to search more.* As a result, moving these workers to other areas where these opportunity costs are higher (higher land rents, lower commuting costs) will induce them to provide higher search levels. "Moving to Opportunity" (MTO) programs are thus the correct policy device to reduce mismatch, rather than lowering search costs in some other way.[19]

There have been several MTO programs implemented in the U.S. The start-

[19] The policy implications would had been quite different if residential segregation had been the result of voluntary choices of workers wishing to share a common culture with their neighbors or to interact in their own language (see among others Akerlof [1], Akerlof and Kranton [2], Ihlanfeldt and Scafidi [19], Selod and Zenou [35] and Battu, McDonald and Zenou [4], who have all emphasized the importance of voluntary choices in the explanation of urban segregation of black workers). If, for example, black workers voluntary want to live together, then it is difficult to move them to predominent white areas. In the present model, in particular in the segregated equilibrium, location choices are decided by comparing short- versus long-term gains and there is no desire to live among similar workers (the extension to black and white workers is straightforward). So workers are ready to move and will then benefit from the policy since it changes the trade offs and induces them to provide higher effort levels.

ing point was the Gautreaux program, implemented in 1976 in the Chicago metropolitan area, which gave housing assistance (i.e. vouchers and certificates) to tenants in order to help diminishing the financial constraints preventing low-income families from relocating to better neighborhoods (Goering, Stebbins and Siewert [14], Turner [40]). Using quasi-experimental methods, the different evaluations of the Gautreaux program suggest that the displaced worker greatly improve their educational as well as their labor market outcomes (Rosembaum [32]). However, one of the main drawback of the Gautreaux program was that blacks were less likely to move because of racial discrimination in the housing market. More recently, the MTO program has be launched by the U.S. Department of Housing and Urban Development (HUD) in Baltimore, Boston, Chicago, Los Angeles and New York since 1994. In these programs, the housing discrimination problem was overcome through the provision of additional services such as housing counseling and landlord outreach. To avoid selection biases, participating families were randomly assigned to one of three groups: (*i*) the 'experimental' or 'MTO' group, which received housing assistance and mobility counseling and was required to move to low-poverty neighborhoods (i.e. tracts with a population poverty rate not exceeding 10%); (*ii*) the 'comparison' or 'Section 8' group, which received housing assistance and could move anywhere; and (*iii*) the 'control' group, which received no vouchers or certificates and could move on their own. The results of this MTO program for most of the five cities mentioned above show a clear improvement of the well-being of participants and better labor market outcomes (Ladd and Ludwig [24], Katz, Kling and Liebman [22], Rosenbaum and Harris [33]).

Our paper is obviously very much in favor of the MTO programs. In light of our results, it predicts that, relative to the 'control' group, displaced workers (from low- to high-rental-housing areas) should provide higher search effort. If labor market participation is a good 'proxy' for search effort, then the findings of Rosenbaum and Harris [33] confirm the predictions of our model. Indeed, using the survey data from the MTO program in Chicago, the findings of these authors, based on interviews an average of 18 months after families moved from public housing to higher rental housing areas, show an increase in labor force participation and employment. More precisely, Rosenbaum and Harris [33] show that: 'After moving to their new neighborhoods, the Section 8 respondents were far more likely to be actively participating in the labor force (i.e. working or looking for a job), while for MTO respondents, a statistically significant increase is evident only for employment per se.'

References

[1] G. Akerlof, Social distance and social decisions, Econometrica 65 (1997) 1005-1027.

[2] G. Akerlof, G. and R. Kranton, Economics and identity, Quarterly Journal of Economics 115 (2000) 715-753.

[3] R. Arnott, Economic theory and the spatial mismatch hypothesis, Urban Studies 35 (1998) 1178-1186.

[4] H. Battu, M. McDonald and Y. Zenou, Do oppositional identities reduce employment for ethnic minorities?, Unpublished manuscript, University of Southampton, 2003.

[5] O.J. Blanchard and P. Diamond, Ranking, unemployment duration, and wages, Review of Economic Studies 61 (1994) 417-434.

[6] M.G. Boarnet and W.T. Bogart, Enterprise zones and employment: evidence from New Jersey, Journal of Urban Economics 40 (1996) 198-215.

[7] J.M. Barron and O. Gilley, Job search and vacancy contacts: Note, American Economic Review 71 (1981) 747-752.

[8] J.K. Brueckner and R.W. Martin, Spatial mismatch: An equilibrium analysis, Regional Science and Urban Economics 27 (1997) 693-714.

[9] J.K. Brueckner and Y. Zenou, Space and unemployment: The labor-market effects of spatial mismatch, Journal of Labor Economics 21 (2003).

[10] R. Chirinko, An empirical investigation of the returns to search, American Economic Review 72 (1982) 498-501.

[11] E. Coulson, D. Laing, D. and P. Wang, Spatial mismatch in search equilibrium, Journal of Labor Economics 19 (2001) 949-972.

[12] M. Fujita, Urban Economic Theory, Cambridge University Press, Cambridge 1989.

[13] L. Gobillon, H. Selod and Y. Zenou, Spatial mismatch: From the hypothesis to the theories, Unpublished manuscript, University of Southampton, 2003.

[14] J. Goering, H. Stebbins and M. Siewert, Promoting Choice in HUD's Rental Assistance Programs, U.S. Department of Housing and Urban Development, Washington, D.C. 1995.

[15] R.E. Hall, A theory of the natural unemployment rate and the duration of employment, Journal of Monetary Economics 5 (1979) 153-169.

[16] H. Holzer, The spatial mismatch hypothesis: what has the evidence shown?, Urban Studies 28 (1991) 105-122.

[17] H. Holzer and J. Reaser, Black applicants, black employees, and urban labor market policy, Journal of Urban Economics 48 (2000) 365-387.

[18] K.R. Ihlanfeldt and D.L. Sjoquist, The spatial mismatch hypothesis: a review of recent studies and their implications for welfare reform, Housing Policy Debate 9 (1998) 849-892.

[19] K.R. Ihlanfeldt and B. Scafidi, Black self-segregation as a cause of housing segregation. Evidence from the multi-city study of urban inequality, Journal of Urban Economics 51 (2002) 366-390.

[20] J.F. Kain, Housing segregation, negro employment, and Metropolitan decentralization, Quarterly Journal of Economics 82 (1968) 175-197.

[21] J. F. Kain, The spatial mismatch hypothesis: Three decades later, Housing Policy Debate 3 (1992) 371-460.

[22] L.F. Katz, J.R. Kling and J.B. Liebman, Moving to opportunity in Boston: Early results of a randomized mobility experiment, Quarterly Journal of Economics 116 (2001) 607-654.

[23] T.C. Koopmans, Representation of preference orderings over time, in C.B. McGuire and R.Radner (Eds.), Decision and Organization, North-Holland, Amsterdam, 1972.

[24] H.F. Ladd and J. Ludwig, Federal housing assistance, residential relocation, and educational opportunities: Evidence from Baltimore, American Economic Review 87 (1997) 272-277.

[25] D.C. Mauer and S.H. Ott, On the optimal structure of government subsidies for entreprise zones and other locational development programs, Journal of Urban Economics 45 (1999) 421-450.

[26] D.T. Mortensen and C.A. Pissarides, New developments in models of search in the labor market, in D. Card and O. Ashenfelter (Eds.), Handbook of Labor Economics, Chap.39, North-Holland, Amsterdam, 1999, pp. 2567-2627.

[27] L. Papke, Tax policy and urban development: evidence from the Indiana enterprise zone program, Journal of Public Economics 54 (1994) 37-49.

[28] C.A. Pissarides, Job matchings with state employment agencies and random search, Economic Journal 89 (1979) 818-833.

[29] C.A. Pissarides, Equilibrium Unemployment Theory, Second edition, M.I.T. Press, Cambridge 2000.

[30] M. Pugh, Barriers to work: the spatial divide between jobs and welfare recipients in metropolitan areas, The Brookings Institution, Washington, D.C. 1998.

[31] C.L. Rogers, Job search and unemployment duration: implications for the spatial mismatch hypothesis, Journal of Urban Economics 42 (1997) 109-132.

[32] J.E. Rosenbaum, Changing the geography of opportunity by expanding residential choice: Lessons from the Gautreaux program, Housing Policy Debate 6 (1995) 231-269.

[33] E. Rosenbaum and L.E. Harris, Residential mobility and opportunities: Early impacts of the Moving to Opportunity demonstration program in Chicago, Housing Policy Debate 12 (2001) 321-346.

[34] J. Seater, Job search and vacancy contacts, American Economic Review 69 (1979) 411-419.

[35] H. Selod. and Y. Zenou, Does city structure affect the labor market outcomes of black workers?, Unpublished manuscript, University of Southampton, 2003.

[36] T.E. Smith and Y. Zenou, Dual labor markets, urban unemployment, and multicentric cities, Journal of Economic Theory 76 (1997) 185-214.

[37] T.E. Smith and Y. Zenou, Spatial mismatch, search effort and workers' location, CEPR Discussion Paper, London, 2002.

[38] T.E. Smith and Y. Zenou, A discrete-time stochastic model of job matching, Review of Economic Dynamics 6 (2003).

[39] S. Turner, Barriers to a better break: Employers discrimination and spatial mismatch in Metropolitan Detroit, Journal of Urban Affairs 19 (1997) 123-141.

[40] M.A. Turner, Moving out of poverty: Expanding mobility and choice through tenant-based assistance, Housing Policy Debate 9 (1998) 373-394.

[41] E. Wasmer and Y. Zenou, Does city structure affect job search and welfare?, Journal of Urban Economics 51 (2002) 515-541.

[42] J. Yinger, Measuring racial discrimination with fair housing audits, American Economic Review 76 (1986) 881-893.

[43] J. Zax and J.F. Kain, Moving to the suburbs: do relocating companies leave their black employees behind?, Journal of Labor Economics 14 (1996) 472-493 .

A. Appendix

A.1. Proof of Proposition 1

Let us first show that there is a unique maximum at each location x. To solve this problem, we begin by partially differentiating (3.7) with respect to s,

$$\frac{\partial}{\partial s}V_0(s,x) = \frac{-a(\alpha+\beta)(b-scx)^{\alpha+\beta-1}R(x)^{-\alpha}cx + \sigma p_h V_1 - V_0(s,x)\sigma p_h}{1-\sigma+\sigma p_h s} \tag{A.1}$$

Hence the first-order condition, $(\partial/\partial s)V_0(s,x) = 0$, is seen to hold iff the numerator is zero, which [by using (2.14)] can be rewritten as

$$\frac{U_0(s,x)(\alpha+\beta)cx}{b-scx} = \sigma p_h \left[V_1 - V_0(s,x)\right] \tag{A.2}$$

or equivalently

$$-\frac{\partial U_0(s,x)}{\partial s} = \sigma p_h \left[V_1 - V_0(s,x)\right] \tag{A.3}$$

To establish the uniqueness of solutions to (A.2) we partially differentiate (A.1) once more [and substitute (A.1) into the result] to obtain:

$$\frac{\partial^2}{\partial s^2}V_0(s,x) = \frac{-aR(x)^{-\alpha}(\alpha+\beta)[1-(\alpha+\beta)](b-scx)^{\alpha+\beta-2}cx - 2\sigma p_h \frac{\partial}{\partial s}V_0(s,x)}{1-\sigma+\sigma p_h s} \tag{A.4}$$

Finally, observing that the sign of (A.4) depends on the numerator, and that the first term in the numerator negative (for positive net incomes) we may conclude that

$$\frac{\partial}{\partial s}V_0(s,x) \geq 0 \;\Rightarrow\; \frac{\partial^2}{\partial s^2}V_0(s) < 0 \tag{A.5}$$

In particular this implies that stationary points of (3.7) can only be local maxima, and thus [by continuity of(A.1)] that there is at most one stationary point. Thus, at each location x there is *at most one solution to* (A.2).

Let us now prove the second part of the proposition.

First, note that in equilibrium this optimal lifetime value must agree with the prevailing lifetime value, V_0, for unemployed workers, i.e., that $V_0(s,x) = V_0$ in (A.2). Note also from (3.1) and (3.3) that in equilibrium we must have

$$U_0(s,x) = (1-\sigma)V_0 - s\sigma p_h \left(V_1 - V_0\right) \tag{A.6}$$

Hence, by substituting these results into (A.2) and solving for s, we obtain

$$s(x) = \frac{\alpha+\beta}{1-(\alpha+\beta)}\left[\frac{b}{(\alpha+\beta)cx} - \frac{(1-\sigma)V_0}{\sigma p_h \left(V_1 - V_0\right)}\right] \tag{A.7}$$

with unique inverse function, $x(s)$, given by (3.9).

In terms of this inverse function (3.9), it follows at once from (A.1) that

$$\frac{\partial}{\partial s} V_0\,(s,x) \gtreqless 0 \Leftrightarrow x \lesseqgtr x(s) \tag{A.8}$$

Let us now prove parts (i), (ii), and (iii) of (3.8). They are established respectively as follows:

- (i) $[x < x(1)]$ Observe from (A.8) and (A.5) that $x < x(1) \Rightarrow \frac{\partial}{\partial s} V_0\,(1,x) > 0 \Rightarrow \frac{\partial^2}{\partial s^2} V_0\,(1,x) < 0$, so that $V_0(\cdot, x)$ must be increasing near $s = 1$. Hence if there is some $s_1 \in [s_0, 1)$ with $V_0(s_1, x) > V_0(1, x)$, then it follows from the continuity of (A.1) that $V_0(\cdot, x)$ must achieve a differentiable minimum at some point interior to $[s_1, 1]$. But since this contradicts (A.5), it follows that no such s_1 can exist, and hence that $V_0(1, x)$ is maximal.

- (ii) $[x > x(s_0)]$ Again by (A.8), $x > x(s_0) \Rightarrow \frac{\partial}{\partial s} V_0\,(s_0,x) < 0$, so that $V_0(\cdot, x)$ must be decreasing near $s = s_0$. Hence if there is some $s_1 \in (s_0, 1]$ with $V_0(s_1, x) > V_0(s_0, x)$, then it again follows from the continuity of (A.1) that $V_0(\cdot, x)$ must achieve a differentiable minimum interior to $[s_0, 1]$, which contradicts (A.5). Thus $V_0(s_0, x)$ must be maximal.

- (iii) $[x(1) \le x \le x(s_0)]$ Finally, it also follows from (A.8) that $x(1) \le x \Rightarrow \frac{\partial}{\partial s} V_0\,(1,x) \ge 0$, and $x \le x(s_0) \Rightarrow \frac{\partial}{\partial s} V_0\,(s_0,x) \le 0$, so that by continuity there is some $s \in [s_0, 1]$ with $\frac{\partial}{\partial s} V_0\,(s,x) = 0$. Hence $s = s(x)$ in (A.7), and we may conclude from the uniqueness of differentiable maxima observed above that $V_0[s(x), x]$ must be maximal.

A.2. Proof of Theorem 2

To establish this result, we introduce the following definitions. With respect to the bid rent functions, R_0 and R_1, a point x is said to be an *intersection point* iff $R_0(x) = R_1(x)$. Next, for any $x \geq 0$ we say that $R_1 \gtreqless R_0$ *to the right of* x iff for some $\varepsilon > 0$ it is true that $R_1(z) \gtreqless R_0(z)$ for all $z \in (x, x+\varepsilon)$. Similarly, for any *positive* point, $x > 0$, we say that $R_1 \gtreqless R_0$ *to the left of* x iff for some $\varepsilon > 0$ it is true that $R_1(z) \gtreqless R_0(z)$ for all $z \in (x-\varepsilon, x)$. An intersection point, $x > 0$, is said to be an *upcrossing* [*downcrossing*] iff $R_1 < R_0$ [$R_1 > R_0$] to the left of x and $R_1 > R_0$ [$R_1 < R_0$] to the right of x. For example, the intersection point x_c in Figure 2 is seen to be an upcrossing, and the point x_p is a downcrossing. Finally, and intersection point, $x > 0$, is said to be a *trivial intersection* iff the curves touch but do not cross at x [i.e., iff for some $\varepsilon > 0$ it is true that either $R_1(z) > R_0(z)$ for all $z \in (x-\varepsilon, x) \cup (x, x+\varepsilon)$, or that $R_1(z) > R_0(z)$ for all $z \in (x-\varepsilon, x) \cup (x, x+\varepsilon)$]. The term 'trivial' here refers to the fact that in an open neighborhood of such intersection points, one bid rent function *strictly* dominates the other except on a set of measure zero. Hence such points can be ignored in characterizing the locational patterns of employed and unemployed workers. From this viewpoint, it is also natural to regard the (possible) intersection point, $x = 0$, as *trivial* whenever it is true that either $R_1 < R_0$ or $R_1 > R_0$ to the right of x. With these definitions, our main result is to show that at equilibrium, R_0 and R_1 can have at most two *nontrivial* intersection points, and that the crossing properties at these points imply the classification in Theorem 2.

To establish this result, we first show that over the range of locations where search intensity is either maximal [$s(x) = 1$] or minimal [$s(x) = s_0$] the bid rent functions R_0 and R_1 satisfy uniform 'relative steepness' conditions which (in terms of the 'crossing' relations) imply that

Lemma 1. *For the bid rent functions R_0 and R_1,*
(i) There is at most one intersection point, $x \leq x(1)$, and if $0 < x < x(1)$, then x must be upcrossing.
(ii) There is at most one intersection point, $x \geq x(s_0)$, and if $x > x(s_0)$, then x must be downcrossing.

Proof: To establish this result we first observe from (4.2) and (4.4) that for the functions S_0 and S_1 defined by

$$S_0(x) \quad = \quad \frac{1}{a} R_0(x)^{\alpha} = \frac{[b - s(x)cx]^{\alpha+\beta}}{(1-\sigma)V_0 - s(x)\sigma p_h (V_1 - V_0)} \tag{A.9}$$

$$S_1(x) \;=\; \frac{1}{a}R_1(x)^\alpha = \frac{(w-cx)^{\alpha+\beta}}{(1-\sigma)V_1+\sigma\rho(V_1-V_0)} \tag{A.10}$$

it follows at once that for all x, $R_0(x) \geq R_1(x) \Leftrightarrow S_0(x) \geq S_1(x)$, and hence that all ordering relations between $R_0(x)$ and $R_1(x)$ are identically that same as those between $S_0(x)$ and $S_1(x)$. This implies in particular that the above intersection and 'crossing' properties defined for R_0 and R_1 are identically the same for S_0 and S_1. Thus we proceed by establishing (i) and (ii) for S_0 and S_1.

(i) Observe that since $s(x) = 1$ for all $x \leq x(1)$, it follows that in this case $S_0(x)$ takes the form

$$S_0(x) = \frac{(b-cx)^{\alpha+\beta}}{(1-\sigma)V_0-\sigma p_h(V_1-V_0)} \tag{A.11}$$

with derivative given by

$$S_0^{'}(x) = -\frac{(\alpha+\beta)c}{b-cx}S_0(x) \tag{A.12}$$

[where $S_0^{'}$ is regarded as a *right* derivative at $x = 0$, and as a *left* derivative at $x = x(1)$]. But since the derivative of $S_1(x)$ is seen to be

$$S_1^{'}(x) = -\frac{(\alpha+\beta)c}{w-cx}S_1(x) \tag{A.13}$$

it follows that if $S_0(x) = S_1(x)$, then by (A.12) and (A.13),

$$-S_0^{'}(x) > -S_1^{'}(x) \Leftrightarrow \frac{1}{b-cx} > \frac{1}{w-cx} \Leftrightarrow w > b \tag{A.14}$$

Finally, since $w > b$ by hypothesis, we may conclude that for all $x \leq x(1)$,

$$S_0(x) = S_1(x) \Rightarrow -S_0^{'}(x) > -S_1^{'}(x) \tag{A.15}$$

and thus that on the interval $[0, x(1)]$, S_0 is *relatively steeper* than S_1 [see for example Fujita (1989, Definition 2.2$'$, p.27)]. To establish property (i), it is enough to observe from (A.15) that each intersection point, $0 < x < x(1)$, must have both $S_1 < S_0$ to the left and $S_1 > S_0$ to the right, and thus must be an upcrossing. Moreover, if there were two intersection points, say $x, z \in [0, x(1)]$ with $x < z$, then (A.15) would also imply that $S_1 > S_0$ to the right of x and $S_1 < S_0$ to the left of z. Hence by continuity there must exist an intermediate intersection point, $w \in (x, z)$, which is not an upcrossing, and hence yields a contradiction. Thus property (i) must hold for (S_0, S_1), and hence for (R_0, R_1)

as well.

(ii) Next observe that since $s(x) = s_0$ for all $x \geq x(s_0)$, it follows in this case that

$$S_0(x) = \frac{(b - s_0 cx)^{\alpha+\beta}}{(1-\sigma)V_0 - s_0\sigma p_h(V_1 - V_0)} \tag{A.16}$$

with derivative given by

$$S_0'(x) = -\frac{(\alpha+\beta)s_0 c}{b - s_0 cx} S_0(x) \tag{A.17}$$

[where S_0' is regarded as a *right* derivative at $x = x(s_0)$]. But if $S_0(x) = S_1(x)$, then by (A.17) and (A.13),

$$-S_1'(x) > -S_0'(x) \Leftrightarrow \frac{1}{w - cx} > \frac{s_0}{b - s_0 cx} \Leftrightarrow w < \frac{b}{s_0} \tag{A.18}$$

which together with condition (4.9) shows that for all $x \geq x(s_0)$,

$$S_0(x) = S_1(x) \Rightarrow -S_1'(x) > -S_0'(x) \tag{A.19}$$

Thus it follows from essentially the same argument as above that property (ii) must hold for (S_0, S_1) and hence for (R_0, R_1). ■

Given these relative steepness results for the regions of constant search intensities, we next analyze the region of *variable* search intensities, namely in the distance band $[x(1), x(s_0)]$. Here the key result is to show that with respect to the functions S_0 and S_1 above:

Lemma 2. *For any pair of intersection points,* $x, z \in [x(1), x(s_0)]$ *with* $x < z$,

$$-S_1'(x) \geq -S_0'(x) \Rightarrow -S_1'(z) > -S_0'(z) \tag{A.20}$$

Proof: We begin by writing $S_0(x)$ in a more explicit form over the range $[x(1), x(s_0)]$ as follows. First by (A.7) we have

$$\begin{aligned} b - s(x)cx &= b - \frac{\alpha+\beta}{1-(\alpha+\beta)}\left[\frac{b}{(\alpha+\beta)cx} - \frac{(1-\sigma)V_0}{\sigma\, p_h\, (V_1 - V_0)}\right]cx \\ &= \left(\frac{\alpha+\beta}{1-(\alpha+\beta)}\right)\left(\frac{(1-\sigma)V_0}{\sigma\, p_h\, (V_1 - V_0)}cx - b\right) \end{aligned} \tag{A.21}$$

and also

$$
\begin{aligned}
&(1-\sigma)V_0 - s(x)\sigma p_h(V_1 - V_0) \\
&= (1-\sigma)V_0 - \frac{b\sigma\, p_h\ (V_1 - V_0)}{[1-(\alpha+\beta)]cx} + \left(\frac{\alpha+\beta}{1-(\alpha+\beta)}\right)(1-\sigma)V_0 \\
&= \frac{1}{1-(\alpha+\beta)}\left[(1-\sigma)V_0 - \sigma\, p_h\ (V_1 - V_0)\,\frac{b}{cx}\right] \qquad \text{(A.22)}
\end{aligned}
$$

so that by (A.9)

$$
S_0(x) = \frac{\left(\frac{\alpha+\beta}{1-(\alpha+\beta)}\right)^{\alpha+\beta}\left(\frac{(1-\sigma)V_0}{\sigma\, p_h\ (V_1-V_0)}cx - b\right)^{\alpha+\beta}}{\frac{1}{1-(\alpha+\beta)}\left[(1-\sigma)V_0 - \sigma\, p_h\ (V_1 - V_0)\,\frac{b}{cx}\right]} \qquad \text{(A.23)}
$$

Next, we introduce certain simplifying notation to facilitate the analysis of this function. If we now let $\theta = \alpha + \beta$, $A = (1-\sigma)V_0$, $D = \sigma p_h\,(V_1 - V_0)$, and let $K_0 = (\theta/D)^{\theta}\,(1-\theta)^{1-\theta}$, then $S_0(x)$ can be written as

$$
S_0(x) = K_0 \frac{cx}{(Acx - bD)^{1-\theta}} \qquad \text{(A.24)}
$$

with derivative

$$
S_0^{'}(x) = \frac{K_0 c}{(Acx - bD)^{1-\theta}} - \frac{K_0 A c^2 x(1-\theta)}{(Acx - bD)^{2-\theta}} \qquad \text{(A.25)}
$$

Similarly, letting $K_1 = 1/[(1-\sigma)V_1 + \sigma\rho\,(V_1 - V_0)]$, it also follows from (A.10) that

$$
S_1(x) = K_1(w - cx)^{\theta} \qquad \text{(A.26)}
$$

with derivative

$$
S_1^{'}(x) = -\frac{c\theta K_1}{(w - cx)^{1-\theta}} \qquad \text{(A.27)}
$$

Hence we have

$$
\begin{aligned}
-S_1^{'}(x) \quad &\geq \quad -S_0^{'}(x) \\
\Leftrightarrow \quad & \frac{c\theta K_1}{(w-cx)^{1-\theta}} \geq \frac{K_0 A c^2 x(1-\theta)}{(Acx - bD)^{2-\theta}} - \frac{K_0 c}{(Acx - bD)^{1-\theta}} \\
\Leftrightarrow \quad & \frac{K_0 A cx(1-\theta)}{(Acx - bD)^{2-\theta}} \leq \frac{\theta K_1}{(w-cx)^{1-\theta}} + \frac{K_0}{(Acx - bD)^{1-\theta}} \qquad \text{(A.28)}
\end{aligned}
$$

But if $S_0(x) = S_1(x)$ then

$$
\begin{aligned}
\frac{\theta K_1}{(w-cx)^{1-\theta}} \quad &= \quad \frac{\theta}{w - cx} K_1 (w - cx)^{\theta} \\
&= \quad \frac{\theta}{w - cx}\left[K_0 \frac{cx}{(Acx - bD)^{1-\theta}}\right] \qquad \text{(A.29)}
\end{aligned}
$$

Thus, by substituting (A.29) into (A.28) and cancelling $K_0/(Acx - bD)^{1-\theta}$, we see that

$$\begin{aligned} -S_1'(x) &\geq -S_0'(x) \\ &\Leftrightarrow (1-\theta)\frac{Acx}{Acx - bD} \leq \theta\frac{cx}{w - cx} + 1 \\ &\Leftrightarrow (1-\theta)\frac{1}{1 - \frac{bD}{Acx}} \leq \theta\frac{1}{\frac{w}{cx} - 1} + 1 \end{aligned} \tag{A.30}$$

Finally, since the left hand side of the last inequality is clearly decreasing in x and the right hand side is increasing in x,[20] it follows that whenever $-S_1'(x) \geq -S_0'(x)$ holds for an intersection point in $[x(1), x(s_0)]$, it must hold as a strict inequality for all subsequent intersection points in $[x(1), x(s_0)]$, thus establishing (A.20). ■

Next we introduce the following additional notation. Let the set of all *intersection points* in $[x(1), x(s_0)]$ be denoted by I, and let the subset of all *nontrivial* intersection points be denoted by I^*. The upcrossing points and downcrossing points in I^* are then denoted respectively by I_U^* and I_D^*. In particular, if x lies in the open interval $(x(1), x(s_0))$, then it follows from the continuity of the derivatives of S_0 and S_1 on $(x(1), x(s_0))$ that whenever $-S_1'(x) \gtrless -S_0'(x)$ for some $x \in (x(1), x(s_0))$, it must be true that $S_1 \gtrless S_0$ to the *left* of x and $S_1 \lessgtr S_0$ to the *right* of x. Conversely, it also follows from continuity of derivatives that if either $S_1 \gtrless S_0$ to the left of x or $S_1 \lessgtr S_0$ to the right of x, then we must have $-S_1'(x) \gtreqless -S_0'(x)$. With these observations and conventions, we next show that

Lemma 3.

(i) *There is at most one point, $x \in I$, with $-S_1'(x) > -S_0'(x)$, and at most one point, $z \in I$, with $-S_1'(z) < -S_0'(z)$;*

(ii) *There is at most one trivial intersection point in $I \cap (x(1), x(s_0))$.*

Proof: (i) In the first case if there are two such points, $x, z \in I$, say with $x < z$, then $-S_1'(x) > -S_0'(x)$ would imply that $S_1 < S_0$ to the right of x, and $-S_1'(z) > -S_0'(z)$ would imply that $S_1 > S_0$ to the left of z. Hence by continuity there must be some upcrossing, $w \in (x, z)$ which by definition must satisfy $-S_1'(w) \leq -S_0'(w)$. But since this together with $x < w$ contradicts

[20] Recall that on the appropriate domains of definition for S_0 and S_1 in the present case, we must have both $Acx > bD$ and $w > cx$.

Lemma 2, it follows that no such w can exist, and hence that no such z can exist. The argument for the second case parallels that of the first, except that the intermediate point, w, must satisfy $-S_1'(w) \geq -S_0'(w)$. But by Lemma 2 this would require that $-S_1'(z) > -S_0'(z)$, and we again obtain a contradiction.

(ii) If $x \in I \cap (x(1), x(s_0))$ is trivial then we must have $S_1'(x) = S_0'(x)$, since a strict inequality would imply a reversal in the ordering of S_1 and S_0 on either side of x. But there cannot be two such points, say $x < z$, since $S_1'(x) = S_0'(x)$ would imply $-S_1'(z) > -S_0'(z)$ by Lemma 2. ■

With these preliminaries, we now have the following intersection properties of R_0 and R_1 on $[x(1), x(s_0)]$:

Lemma 4. *The bid rent functions R_0 and R_1 exhibit the following three properties on $[x(1), x(s_0)]$:*
(i) There are at most two nontrivial intersection points in $[x(1), x(s_0)]$.
(ii) If the first nontrivial intersection point is a downcrossing, then it is the only intersection point in $[x(1), x(s_0)]$.
(iii) If there are two nontrivial intersection points in $[x(1), x(s_0)]$, then the first is upcrossing and the second is downcrossing.

Proof: To establish this result, we again argue in terms of the functions S_0 and S_1. First observe from the continuity of S_0 and S_1 that the set I of all intersection points in $[x(1), x(s_0)]$ is closed, and hence compact. Thus if I is nonempty there must exist a *first* intersection point, x_1, in $[x(1), x(s_0)]$. In terms of x_1 and the sets, I^*, I_U^*, I_D^*, defined above, our objective is thus to show that (i) $|\,I^*\,| \leq 2$, (ii) $x_1 \in I_D^* \Rightarrow I^* = \{x_1\}$, and (iii) $I^* = \{x_1, x_2\} \Rightarrow [x_1 \in I_U^*, x_2 \in I_D^*]$. To do so, we proceed by considering each possible relation between the derivatives S_1' and S_0' at x_1, and show that conditions $[(i), (ii), (iii)]$ hold (possibly vacuously) in each case. Because the derivatives S_1' and S_0' are possibly discontinuous at the end points $x(1)$ and $x(s_0)$, it is convenient to treat these cases separately. Hence for the present we assume that x_1 lies in the open interval $(x(1), x(s_0))$, so that right and left derivatives at x_1 are the same:

(a) First if $-S_1'(x_1) > -S_0'(x_1)$, then it follows at once from Lemmas 2 and 3(i) that $I^* = \{x_1\}$, and hence that (ii) holds [and (i),(iii) hold vacuously].

(b) If $S_1'(x_1) = S_0'(x_1)$, then by Lemma 2 it follows that any subsequent point, $x_2 \in I$, must satisfy $-S_1'(x_2) > -S_0'(x_2)$. But by Lemma 3(i) there is at most one such point, so that $I = \{x_1, x_2\}$. Hence, suppose first that x_1 is trivial (i.e., $x_1 \in I - I^*$). Then $I^* = \{x_2\}$, so that again (ii) holds [and (i),(iii) hold vacuously]. Now suppose that x_1 is nontrivial (i.e., $x_1 \in I^*$). In this case we see that $I^* = \{x_1, x_2\}$, and $-S_1'(x_2) > -S_0'(x_2) \Rightarrow x_2 \in I_D^*$. Hence if it can be shown $x_1 \in I_U^*$, then it will follow that (i) and (iii) both hold [and that (ii) holds vacuously]. To do so, observe that since $S_1'(x_1) = S_0'(x_1)$ also implies from Lemmas 2 and 3(i) that $I \cap (x_1, x_2) = \emptyset$, and since $x_2 \in I_D^* \Rightarrow S_1 > S_0$ to the left of x_2, it follows that in this case $S_1 > S_0$ must hold on all of (x_1, x_2). Thus $S_1 > S_0$ to the right of x_1, which together with the definition of nontrivial points, implies that $S_1 > S_0$ cannot hold to the left of x_1. But if $S_1 < S_0$ to the left of x_1, then $x_1 \in I_U^*$, and we are done. So it remains only to rule out the case in which both $S_1 < S_0$ and $S_1 > S_0$ fail to hold to the left of x_1. To do so, observe that both these conditions can fail only if there is an increasing sequence (x_n) in $[x(1), x(s_0)]$ converging to x_1 with $S_1(x_n) = S_0(x_n)$ for all n. But by definition, all such points must lie in I, which contradicts the definition of x_1 (as the first point in I). Hence no such sequence can exist, and it must be true that $x_2 \in I_D^*$.

(c) Finally, suppose that $-S_1'(x_1) < -S_0'(x_1)$, so that in particular, $x_1 \in I_U^*$. If there is a second point, $x_2 \in I^*$, then we may assume that x_2 is the first nontrivial point to the right of x_1. [Indeed, since I is closed and since Lemma 3(ii) implies the existence of at most one trivial point in $I \cap (x_1, x(s_0))$, nonexistence of a first point in $I \cap (x_1, x(s_0))$ would entail the existence of a decreasing sequence in I^* converging to x_1, which would contradict the condition that $S_1 > S_0$ to the right of x_1]. But since $S_1 > S_0$ to the right of x_1, it then follows that $S_1 > S_0$ must also hold to the left of x_2 [since there is at most one trivial intersection point, $z \in (x_1, x_2)$, and since $S_1 > S_0$ to the left of z implies $S_1 > S_0$ to the right of z]. But this in turn implies that $-S_1'(x_2) \geq -S_0'(x_2)$. Thus, by replacing x_1 with x_2 in part (a), it now follows from part (a) that if $-S_1'(x_2) > -S_0'(x_2)$, then both conditions (i) and (iii) hold with $I^* = \{x_1, x_2\}$ [and (ii) holds vacuously]. Moreover, if $S_1'(x_2) = S_0'(x_2)$, then by the same argument as in part (b) above, it follows that there is at most one additional point, $x_3 \in I$, beyond x_2, and that $S_1 > S_0$ must hold on all of (x_2, x_3). But this together with $S_1 > S_0$ on all of (x_1, x_2) would then contradict the hypothesis that x_2 is nontrivial, so that no intersection points can exist beyond x_2. Hence the only remaining possibility is that $S_1 < S_0$ to the right of x_2, which implies that $x_2 \in I_D^*$. Thus both conditions (i) and (iii) again hold with $I^* = \{x_1, x_2\}$ [and (ii) holds vacuously].

The last remaining possibility is that $x_1 = x(1)$ or $x_1 = x(s_0)$. But since

$x(s_0)$ is the *last* point in $[x(1), x(s_0)]$, it follows that the condition, $x_1 = x(s_0)$, can never contradict $(i), (ii)$,or (iii), so this case can be dispensed with. Turning finally to the case, $x_1 = x(1)$, observe that if $S_0'(x_1)$ is interpreted as a *right* derivative, then (A.25) and continues to hold. Hence Lemmas 2 and 3 are still applicable to x_1, and it follows that all arguments in $(a), (b)$,and (c) not involving properties to the left side of x_1 must continue to hold. An inspection of these conditions shows that the only case which needs to be reconsidered is the case in (b) with $x_1 \in I^*$ and $S_1'(x_1) = S_0'(x_1)$. If there is a second point, $x_2 \in I$, then by Lemmas 2 and 3 this point is unique and satisfies $-S_1'(x_2) > -S_0'(x_2)$, so that $I^* = \{x_1, x_2\}$ and $x_2 \in I_D^*$. Hence it must again be true that $S_1 > S_0$ on all of (x_1, x_2). But recall also that if $S_0'(x_1)$ is regarded as a *left* derivative then (A.15) continues to hold at $x = x(1)$. Hence the left derivatives at x_1 must satisfy $-S_1'(x_1) < -S_0'(x_1)$, so that $S_1 < S_0$ to the left of x_1. Finally, this together with $S_1 > S_0$ to the right of x_1 implies that $x_1 \in I_U^*$, so that conditions (i) and (iii) again hold [with (ii) holding vacuously]. ■

Given these partial results, we are now ready to consider the full set of possible intersections between R_0 and R_1 over the range where they are both positive, which is contained in the interval $[0, \frac{w}{c}]$. In a manner paralleling the notation for local analysis of $[x(1), x(s_0)]$ in Lemma 4 above, we now denote the (possibly larger) set of intersection points in $[0, \frac{w}{c}]$ by $\mathbf{I}$, with corresponding subsets $\mathbf{I}^*, \mathbf{I}_U^*, \mathbf{I}_D^*$ of nontrivial intersections, upcrossings, and downcrossings. Given this notation, we first show that these definitions yield a complete classification of nontrivial intersection points

Lemma 5.

(i) *There are at most five intersection points, i.e.,* $|\,\mathbf{I}\,| \leq \mathbf{5}$.

(ii) *Every positive nontrivial intersection point is either an upcrossing or a downcrossing, i.e.,* $\mathbf{I}^* = \mathbf{I}_U^* \cup \mathbf{I}_D^*$.

Proof: (i) By Lemma 1 there is at most one intersection point in $\mathbf{I} \cap [0, x(1)]$ and at most one in $\mathbf{I} - [0, x(s_0))$. Moreover, by part (ii) of Lemma 3 together with part (i) of Lemma 4, there are at most three points in $\mathbf{I} \cap (x(1), x(s_0))$, making a total of at most five. (ii) If $x \in \mathbf{I}^* - (\mathbf{I}_U^* \cup \mathbf{I}_D^*)$, then either both conditions $R_0 < R_1$ and $R_0 > R_1$ must fail to the left of x or both must fail to the right of x. But in either case, there must be a sequence of distinct intersections, (x_n) in $\mathbf{I} - \{x\}$ converging to x, which violates part (i) above. Hence $\mathbf{I}^* - (\mathbf{I}_U^* \cup \mathbf{I}_D^*) = \emptyset$. ■

By employing these properties, we can now establish our main result, which amounts to a sharper version of Theorem 2 in the text. In particular, property (i) below implies that there are at most two reversals in the ordering of R_0 and R_1, and hence at most three distinct rings of locators. Property (ii) then implies that if there are two reversals, say at points x_c and x_1, then $R_0 > R_1$ must hold almost everywhere on both end intervals, $(0, x_c), (x_1, \frac{w}{c})$, and $R_1 > R_0$ must hold almost everywhere on the middle interval, (x_c, x_1).[21] Hence the central core and peripheral ring can only be occupied by unemployed workers, and the middle ring by employed workers.[22]

Theorem A.1 (Classification of Intersections). *The rents functions, R_0 and R_1, exhibit the following two properties on the interval $[0, \frac{w}{c}]$:*
(i) *There are at most two nontrivial intersection points.*
(ii) *If there are two nontrivial intersection points, then the first is upcrossing and the second is downcrossing.*

Proof: In terms of the above notation, these two properties can be equivalently stated as follows: (i) $|\mathbf{I}^*| \leq 2$, and (ii) $\mathbf{I}^* = \{x_1, x_2\} \Rightarrow [x_1 \in \mathbf{I}_U^*, x_2 \in \mathbf{I}_D^*]$. To establish these properties we consider two cases, depending on the ordering of rents at $x = 0$:

(a) Suppose first that $R_1(0) < R_0(0)$, and that $\mathbf{I} \neq \emptyset$ with first intersection point $x_1 \in \mathbf{I}$ [which exists by Lemma 5(i) above]. Then by definition we must have $R_1 < R_0$ to the left of x_1. Here there are three possibilities to consider, depending on the location of x_1:

(a_1) If $x_1 < x(1)$, then by Lemma 1(i) it follows that $x_1 \in \mathbf{I}_U^*$ and that $\mathbf{I} \cap [0, x(1)] = \{x_1\}$. Hence if there is a second point, $x_2 \in \mathbf{I}$, then $x_2 > x(1)$, and we may again assume that x_2 is the first such point. This implies that $R_1 > R_0$ on (x_1, x_2), and in particular, that $R_1 > R_0$ to the left of x_2. But if $x_2 \geq x(s_0)$, then by Lemma 1(ii) it will follow that $\mathbf{I} = \mathbf{I}^* = \{x_1, x_2\}$ and that $x_2 \in \mathbf{I}_D^*$ whenever $x_2 > x(s_0)$. Similarly, if $x_2 = x(s_0)$, then since (A.19) was shown to hold for this case, it follows that $R_1 < R_0$ to the right of x_2. But this together with $R_1 > R_0$ to the left of x_2, again implies that $x_2 \in \mathbf{I}_D^*$.

[21] In fact Lemma 5(i) implies that 'almost everywhere' can be sharpened to 'at most three trivial intersection points'.

[22] Note that we here ignore the agricultural rent level, R_A. Hence the existence of an upcrossing point followed by a downcrossing point in $[0, \frac{w}{c}]$ need not imply the existence of both a core and peripheral ring of unemployed in equilibrium. In particular, if $R_1 < R_A$ at the downcrossing point, then there will only be a core ring of unemployed. However, this possibility does not conflict with the assertion of Theorem 2.

Hence conditions $[(i),(ii)]$ will hold whenever $x_2 \geq x(s_0)$, so that we may now assume $x(1) < x_2 < x(s_0)$. Here there are two cases:

(a_{11}) Suppose first that x_2 is nontrivial. Then since $R_1 > R_0$ to the left of x_2 implies that x_2 is not in $\mathbf{I}_U^*$, it follows from Lemma 5(ii) that $x_2 \in \mathbf{I}_D^*$. Hence if there were a next intersection point, say $x_3 \in \mathbf{I}^*$, then since x_2 is the only point in I by Lemma 4(ii), we would have $x_3 > x(s_0)$. Moreover, since $R_1 < R_0$ to the right of x_2, it would also follow that $R_1 < R_0$ on (x_2, x_3), so that in particular, x_3 could not be downcrossing. But since this contradicts Lemma 1(ii), we may conclude that no such point exists, and hence that $\mathbf{I}^* = \{x_1, x_2\}$ with $x_1 \in \mathbf{I}_U^*$ and $x_2 \in \mathbf{I}_D^*$. Thus conditions $[(i),(ii)]$ must hold in this case.

(a_{12}) Next suppose that x_2 is trivial. Then by the proof of Lemma 3(ii) it follows not only that x_2 is the unique trivial intersection point in $(x(1), x(s_0))$, but also that $S_0'(x_2) = S_1'(x_2)$. Hence by Lemma 2 there can be at most one subsequent nontrivial point in I, and it must be downcrossing. Moreover, by Lemma 1(ii), there is at most one point in $\mathbf{I} \cap (x(s_0), \frac{w}{c}]$, which must also be downcrossing. Finally since the occurrence of both these possibilities would entail the existence of an intermediate upcrossing point which would then violate either Lemma 2 or Lemma 1(ii), it follows that there can be at most one point in $\mathbf{I} \cap (x_2, \frac{w}{c}]$ and that this point, say x_3, must be downcrossing. Hence if such an x_3 exists, then conditions $[(i),(ii)]$ hold with $\mathbf{I}^* = \{x_1, x_3\}$, $x_1 \in \mathbf{I}_U^*$ and $x_3 \in \mathbf{I}_D^*$. If no such point exists, then condition (i) holds with $\mathbf{I}^* = \{x_1\}$ and condition (ii) holds vacuously.

(a_2) If $x_1 \in [x(1), x(s_0)]$, then by Lemma 4 there are only two cases to consider, depending on whether $|I^*| = 1$ or $|I^*| = 2$.

(a_{21}) If $I^* = \{x_1\}$, and if there is a second intersection, $x_2 \in \mathbf{I}$, then we must have $x_2 > x(s_0)$, so that $x_2 \in \mathbf{I}_D^*$ by Lemma 1(ii). But since $R_1 < R_0$ to the left of x_1 precludes the possibility that $x_1 \in \mathbf{I}_D^*$, it follows from Lemma 5(ii) that $x_1 \in \mathbf{I}_U^*$, and hence that conditions $[(i),(ii)]$ hold with $\mathbf{I}^* = \{x_1, x_2\}$.

(a_{22}) If $I^* = \{x_1, x_2\}$, and if there were a third intersection, $x_3 \in \mathbf{I}$, then we would again have $x_3 > x(s_0)$, so that $x_3 \in \mathbf{I}_D^*$ by Lemma 1(ii). But since $x_2 \in \mathbf{I}_D^*$ by Lemma 4(iii), this would in turn imply the existence of an intermediate upcrossing point, $x \in (x_2, x_3)$, which would again contradict either Lemma 2 or Lemma 1(ii), depending on whether $x \leq x(s_0)$ or $x > x(s_0)$. Hence no such point can exist, and conditions $[(i),(ii)]$ hold with $\mathbf{I}^* = \{x_1, x_2\}$.

(a_3) Finally we note that $x_1 > x(s_0)$ is not possible. For since $x_1 \notin \mathbf{I}_D^*$ [as observed in (a_{21}) above], this would contradict Lemma 1(ii).

(b) Next suppose that $R_1(0) \geq R_0(0)$, and that $\mathbf{I} \neq \emptyset$ with first intersection point $x_1 \in \mathbf{I}$. If $R_1(0) = R_0(0)$ then $x_1 = 0$ and $R_1 > R_0$ to the right of x_1[by (A.15)]. Thus by our conventions above, x_1 is a trivial intersection point and can be discarded. Moreover, the second intersection point, say x_1' must have $R_1 > R_0$ to the left. Hence we may assume without loss of generality that $R_1(0) > R_0(0)$ and $R_1 > R_0$ to the left of x_1. This in turn implies that $x_1 \notin \mathbf{I}_U^*$, and hence from (A.15) that $x_1 > x(1)$. Moreover, if $x_1 \geq x(s_0)$, then since $R_1 > R_0$ to the left of x_1, it follows by the same argument as for the case ‘ $x_2 \geq x(s_0)$’ in (a_1) above, that $x_1 \in \mathbf{I}_D^*$ and hence that conditions (i) holds with $\mathbf{I}^* = \{x_1\}$ and condition (ii) holds vacuously. Thus we may now assume that $x(1) < x_1 < x(s_0)$. But again, $R_1 > R_0$ to the left of x_1 implies that this case for x_1 is formally identical to that for x_2 in (a_{11}) and (a_{12}) above. Hence if x_1 is nontrivial, it now follows from the argument in case (a_{11}) that condition (i) holds with $\mathbf{I}^* = \{x_1\}$ and (ii) holds vacuously. Similarly, if x_1 is trivial, then the argument in case (a_{12}) now shows that either there is an additional point, $x_2 \in \mathbf{I}_D^* \cap (x_1, \frac{w}{c}]$ with $\mathbf{I}^* = \{x_2\}$, or we must have $\mathbf{I}^* = \emptyset$. In either case, conditions (i) holds and (ii) holds vacuously.

Thus we may conclude that conditions $[(i), (ii)]$ hold in all cases, and the result is established. ■

A.3. Existence and uniqueness of Core-Periphery equilibria

Theorem 5.1 shows the uniqueness of the core-periphery (CP) equilibrium whereas Theorem 5.2 demonstrates its existence.

In order to prove these two theorems, we first need to introduce the concept of 'semi-equilibrium', which is a weaker concept than our CP equilibrium, and to show that for each choice of employment level, N_1, this semi-equilibrium exists uniquely. Then, using this first result, we prove the uniqueness (Theorem 5.1) and the existence (Theorem 5.2) of the core-periphery (CP) equilibrium.

A.3.1. Existence and uniqueness of Semi-Equilibria

To analyze the existence and uniqueness of *CP*-equilibria, we begin by introducing a weaker class of 'semi-equilibria' which will be :

Definition 3. *For any subvector of admissible parameters,* $\widetilde{\theta} = (\rho, \gamma, s_0, \lambda, N, \sigma, \alpha, \beta, b, w, c)$*, a vector of values,* $\xi\left(\widetilde{\theta}\right) = (N_0^c, N_0^p, N_1, p_h, U_0^c, U_0^p, U_1, x_c, x_p)$*, is said to be* a semi-equilibrium for $\widetilde{\theta}$ *iff conditions [(5.15),(5.18),(5.19),(5.23), (5.24),(5.28), (5.29),(5.30)] are satisfied.*

Such semi-equilibria thus satisfy all conditions of *CP*-equilibria except the peripheral unemployment condition, (5.25), and the three boundary-point conditions [(5.31),(5.32),(5.33)]. Notice in particular that since both the frontier location, x_f, and the agricultural rent, R_A, appear only in the missing equations (5.25) and (5.33), they are not defined in semi-equilibria. Hence the parameter subvector, $\widetilde{\theta}$, excludes R_A, and the semi-equilibrium vector, $\xi(\widetilde{\theta})$, excludes x_f.

To establish existence and uniqueness of semi-equilibria for each N_1, we shall construct an appropriate mapping which will be shown to have a unique fixed point. To do so, observe first that there is only a bounded range of employment levels, N_1, which are possible in equilibrium. In particular, (5.26) shows that mean search intensity, $\overline{s}$, must always lie between s_0 and 1. But since u is monotone decreasing in $\overline{s}$ by Theorem 1, and since (5.27) and (5.29) show that $N_1 = N(1-u)$, it then follows that N_1 is monotone increasing in $\overline{s}$. Hence if $u_{\min} = \psi^{-1}(1)$, $u_{\max} = \psi^{-1}(s_0)$, and we let

$$N_1^{\min} = N(1-u_{\max}) = N[1-\psi^{-1}(s_0)] \tag{A.31}$$

$$N_1^{\max} = N(1-u_{\min}) = N[1-\psi^{-1}(1)] \tag{A.32}$$

then in equilibrium we must have

$$N_1^{\min} \leq N_1 \leq N_1^{\max} \tag{A.33}$$

Given these restrictions, observe next from (5.26) that labor market steady states depend only on the subvector of parameters, $\overline{\theta} = (\rho, \gamma, s_0, \lambda, N)$, which we now designate as the *steady-state parameters*. With these definitions, we now have the following characterization of labor market 'core-periphery' steady states with employment levels satisfying (A.33):

Proposition 3. *For each vector of steady-state parameters, $\overline{\theta} = (\rho, \gamma, s_0, \lambda, N)$, and each employment level, $N_1 \in [N_1^{\min}, N_1^{\max}]$, there is a unique set of steady-state values $[N_0^c(N_1), N_0^p(N_1), p_h(N_1)]$. In addition:*
(i) the ratio, $N_0^c(N_1)/N_1$, is increasing in N_1, and
(ii) the remaining functions, $N_0^p(N_1)$ and $p_h(N_1)$, are decreasing in N_1.

Proof. If for any given $\overline{\theta} = (\rho, \gamma, s_0, \lambda, N)$ and $N_1 \in [N_1^{\min}, N_1^{\max}]$, we let $\overline{s}(N_1) = \psi\left(\frac{N-N_1}{N}\right) \in [s_0, 1]$, and solve for (N_0^c, N_0^p) using [(5.28),(5.29)], we obtain the unique (nonnegative) values

$$N_0^c(N_1) = \frac{\overline{s}(N_1) - s_0}{1 - s_0}(N - N_1) \tag{A.34}$$

$$N_0^p(N_1) = \frac{1 - \overline{s}(N_1)}{1 - s_0}(N - N_1) \tag{A.35}$$

which together yield a unique hiring probability, $p_h(N_1) \in (0,1)$, by (5.30).

To establish that $N_0^c(N_1)/N_1$ is increasing in N_1, it is convenient to rewrite (5.28) as

$$\begin{aligned} \psi(u) &= \frac{N_0^c + s_0(N - N_1 - N_0^c)}{N - N_1} \\ &= \frac{(N_0^c/N)(1 - s_0) + s_0 u}{u} \end{aligned} \tag{A.36}$$

which in turn implies that, N_0^c, can be written in terms of u as follows:

$$N_0^c(u) = \frac{N}{1 - s_0}\left[u\psi(u) - s_0 u\right] \tag{A.37}$$

Hence observing that $N_1/N = 1 - u$, it follows that the ratio, $R_c \equiv N_0^c/N_1$, can also be written in terms of u as

$$R_c(u) = \frac{1}{1 - s_0}\left[\frac{u\psi(u)}{1 - u} - s_0 \frac{u}{1 - u}\right] \tag{A.38}$$

But since u decreasing in N_1 by (5.27) it then suffices to show that $R_c(u)$ is decreasing. Moreover, since the second term in brackets is decreasing in u, it is enough to show that the function,

$$f_1(u) = \frac{u\psi(u)}{1-u} \tag{A.39}$$

is also decreasing in u. But by (2.6) it follows that

$$\begin{aligned} f_1(u) &= -\frac{u}{1-u}\left\{\left(\frac{u+d}{\gamma u}\right)\ln\left[1-\left(\frac{\rho}{1-\rho}\right)\left(\frac{1-u}{u+d}\right)\right]\right\} \\ &= -\frac{1}{\gamma}\left(\frac{u+d}{1-u}\right)\ln\left[1-\left(\frac{\rho}{1-\rho}\right)\left(\frac{1-u}{u+d}\right)\right] \\ &= -\frac{1}{\gamma}\frac{\ln[1-\omega x(u)]}{x(u)} \end{aligned} \tag{A.40}$$

where $\omega = \rho/(1-\rho)$, and where the function, $x(u) = (1-u)/(u+d) \in (0,1)$, is decreasing in u. Hence it follows that

$$f_1'(u) = \frac{x'(u)}{\gamma x(u)^2}\left\{\frac{\omega x(u)}{1-\omega x(u)} + \ln\left[1-\omega x(u)\right]\right\} \tag{A.41}$$

which together with $x'(u) < 0$, implies that we need only show that the quantity in brackets is positive. But if we let $z = \omega x \in (0,1)$ and consider the function,

$$f_2(z) = \ln\left[1-z\right] + \frac{z}{1-z} \tag{A.42}$$

then

$$f_2'(z) = \frac{z}{(1-z)^2} > 0 \tag{A.43}$$

together with the boundary condition, $f_2(0) = 0$, implies that $f_2(z) > 0$ for all $z \in (0,1)$, and hence that $f_1(u)$ is decreasing.

To establish (ii), observe first that since $N_0^c(N_1)$ is increasing in N_1 by (i), it follows at once from the accounting identity (5.29) that $N_0^p(N_1)$ is decreasing in N_1. To show that $p_h(N_1)$ is also decreasing in N_1, recall from (2.2) and (2.6) that we have the steady-state identity

$$\begin{aligned} \rho\,(1-u) &\equiv (1-\rho)\; u\psi(u)p_h(u) \\ &\Rightarrow \frac{\rho}{1-\rho} \equiv f_1(u)p_h(u) \end{aligned} \tag{A.44}$$

where $f_1(u)$ is given by (A.39) above. But since $f_1(u)$ was shown to be decreasing, this implies (A.44) that $p_h(u)$ must be increasing, and hence that $p_h(N_1)$ is decreasing in N_1 ■

Result (i) is perhaps most surprising, for it tells us that among those labor-market steady states which are consistent with core-periphery land use patterns, higher employment levels, N_1, can only be supported by increasing the number of full-time job searchers *per employee*. This appears to be a consequence of both market friction effects and the fact that all other job searchers use only minimal search intensity, s_0. Note also that since higher employment levels decrease the hiring probability, $p_h(N_1)$, this will tend to discourage additional full-time searchers. Hence these two conflicting forces already suggest that there should be at most one CP-equilibrium.

Using this result, we can construct the desired mapping as follows. First, if the utility ratios in (5.18) and (5.19) are denoted respectively by $r_c = U_0^c/U_1$ and $r_p = U_0^p/U_1$, then it follows from (5.15) that

$$r_p = \tau(p_h)r_c + [1 - \tau(p_h)] \tag{A.45}$$

Moveover, since r_c is seen from (5.18) to be a function of x_c:

$$r_c(x_c) = \left(\frac{b - cx_c}{w - cx_c}\right)^{\alpha+\beta} \tag{A.46}$$

and since Proposition 3 shows that the steady state value of p_h is a well defined function of N_1, it follows that we may express the equilibrium value of r_p as a function of (N_1, x_c):

$$r_p(N_1, x_c) = \tau[p_h(N_1)] \cdot r_c(x_c) + \{1 - \tau[p_h(N_1)]\} \tag{A.47}$$

Hence if (5.19) is solved for x_p in terms of r_p, then we may also write x_p as a function of (N_1, x_c):

$$x_p(N_1, x_c) = \frac{w \cdot r_p(N_1, x_c)^{\frac{1}{\alpha+\beta}} - b}{c\left(r_p(N_1, x_c)^{\frac{1}{\alpha+\beta}} - s_0\right)} \tag{A.48}$$

Next observe that if the utility parameters in the population densities (5.20), (5.21), and (5.22) are now viewed as variables by writing $\eta_1(x, U_1)$, $\eta_0^c(x, U_0^c)$ and $\eta_0^p(x, U_0^p)$, then it follows by inspection of (5.23), (5.24), and (5.25) that the right hand sides are positive decreasing functions of their utility variables, which range from zero to infinity as the utilities range from infinity to zero. In particular, since the value, $N_0^c(N_1) \in (0, N_1)$, is well defined by (A.34), it then follows from (5.23) that there must be a unique utility value, $U_0^c(N_1, x_c)$, satisfying the condition that

$$N_0^c(N_1) = \int_0^{x_c} \eta_0^c[x, U_0^c(N_1, x_c)]dx \tag{A.49}$$

Hence (A.49) implicitly defines the function $U_0^c(N_1, x_c)$ [which can be made explicit by solving for U_0^c in (5.25)]. By combining this with (A.46), we may also express U_1 as a function of (N_1, x_c):

$$U_1(N_1, x_c) = \frac{U_0^c(N_1, x_c)}{r_c(x_c)} \tag{A.50}$$

Finally, by employing these functions, we may define the desired mapping, $\phi(N_1, x_c)$, as follows:

$$\phi(N_1, x_c) = \int_{x_c}^{x_p(N_1, x_c)} \eta_1[x, U_1(N_1, x_c)]dx \tag{A.51}$$

By comparing (A.49) with (5.24), we see that ϕ essentially defines a new value of N_1 for each given pair (N_1, x_c). Given this mapping, our key result is to establish the following fixed-point property of ϕ:

Proposition 4. *For each* $N_1 \in (N_1^{\min}, N_1^{\max}]$ *there exists a unique* $x_c > 0$ *such that*

$$\phi(N_1, x_c) = N_1 \tag{A.52}$$

Proof. To establish the desired result, observe from (A.52) that it suffices to show that the function $\phi(N_1, \cdot)$ is decreasing with

$$\lim_{x_c \to 0} \phi(N_1, x_c) = \infty \tag{A.53}$$

and

$$\lim_{x_c \to b/c} \phi(N_1, x_c) = 0 \tag{A.54}$$

For this will imply that $\phi(N_1, \cdot)$ takes on the value N_1 exactly once. To establish that $\phi(N_1, \cdot)$ is decreasing, observe from (A.51) together with the decreasing monotonicity of $\eta_1(x, U_1)$ in U_1, that it is enough to show that (i) $x_p(N_1, x_c)$ is decreasing in x_c, and (ii) $U_1(N_1, x_c)$ is increasing in x_c. For then it will follow (by inspection) that the integral on the right hand side of (A.51) must diminish as x_c increases. To establish (i) observe first that $r_c(x_c)$ is decreasing in x_c (since $b < w$). But since $p_h(N_1)$ is fixed (for the given value of N_1) this in turn implies from (A.47) that $r_p(N_1, x_c)$ is decreasing in x_c. Moreover, since (A.48) shows that

$$\begin{aligned} \frac{\partial}{\partial r_p} x_p(r_p) > 0 \quad &\Leftrightarrow \quad w > \left(\frac{w r_p^{\frac{1}{\alpha+\beta}} - b}{c r_p^{\frac{1}{\alpha+\beta}} - c s_0} \right) c \\ &\Leftrightarrow \quad c w s_0 < bc \\ &\Leftrightarrow \quad w < \frac{b}{s_0} \end{aligned} \tag{A.55}$$

we see from (4.9) that x_p must be increasing r_p, so that $x_p(N_1, x_c)$ is also decreasing in x_c, and (i) must hold. To establish (ii), observe next that since $N_0^c(N_1)$ is fixed, it follows from (A.49) that an increase in x_c implies that $\eta_0^c[\cdot, U_0^c(N_1, x_c)]$ must decrease, and hence that $U_0^c(N_1, x_c)$ must be increasing in x_c. Finally, since $r_c(x_c)$ is decreasing, this in turn implies from (A.50) that $U_1(N_1, x_c)$ must be increasing in x_c, so that (ii) holds as well.

Next, to establish (A.53), observe from (A.51) that it suffices to show that (i) $\lim_{x_c \to 0} x_p(N_1, x_c) = \overline{x}_p > 0$, and (ii) $\lim_{x_c \to 0} U_1(N_1, x_c) = 0$. For then it will follow both that the limiting interval of integration in (A.51) has positive measure, and that the integrand, $\eta_1[\cdot, U_1(N_1, x_c)]$, diverges to infinity, implying that $\phi(N_1, x_c)$ must also diverge to infinity. To establish (i), note first from (A.46) that $x_c \to 0$ implies $r_c(x_c) \to (b/w)^{\alpha+\beta} \in (0, 1)$. But since this in turn implies from (A.47) that $r_p(N_1, x_c) \to \tau_1 (b/w)^{\alpha+\beta} + (1 - \tau_1) > (b/w)^{\alpha+\beta}$ [where $\tau_1 = \tau[p_h(N_1)] > 0$], and since x_p was shown above to be increasing in r_p, it then follows from (A.48) and (4.9) that $\lim_{x_c \to 0} x_p(N_1, x_c) = \overline{x}_p$ with

$$\overline{x}_p > \frac{w\left[(b/w)^{\alpha+\beta}\right]^{\frac{1}{\alpha+\beta}} - b}{c\left(\left[(b/w)^{\alpha+\beta}\right]^{\frac{1}{\alpha+\beta}} - s_0\right)} = \frac{0}{\frac{cs_0}{w}\left(\frac{b}{s_0} - w\right)} = 0 \tag{A.56}$$

To establish (ii), observe from the constancy of $N_0^c(N_1)$ together with (A.49) that

$$\begin{aligned} x_c \to 0 \ &\Rightarrow \ \eta_0^c[\cdot, U_0^c(N_1, x_c)] \to \infty \\ &\Rightarrow \ U_0^c(N_1, x_c) \to 0 \end{aligned} \tag{A.57}$$

But since $r_p(N_1, x_c)$ was shown above to have a finite positive limit as $x_c \to 0$, we may then conclude from (A.50) that $\lim_{x_c \to 0} U_1(N_1, x_c) = 0$, and (ii) hold as well.

Finally to establish (A.54), observe first from (A.46) and (A.47) that $b < w \Rightarrow r_p < 1 \Rightarrow x_p < (w - b)/[c(1 - s_0)]$, so that the limiting interval of integration in (A.51) must be bounded as $x_c \to b/c$. Hence it suffices to show that

$$\lim_{x_c \to b/c} U_1(N_1, x_c) = \infty \tag{A.58}$$

For this will in turn imply that the integrand, $\eta_1[\cdot, U_1(N_1, x_c)]$, converges uniformly to zero, and hence that $\lim_{x_c \to b/c} \phi(N_1, x_c) = 0$. To establish (A.58), observe first from (A.46) that $x_c \to b/c \Rightarrow r_c(x_c) \to 0$, and hence from (A.50) that either (A.58) holds or $U_0^c(N_1, x_c) \to 0$. To see that the latter is not possible, observe that this would imply $\eta_0^c[\cdot, U_0^c(N_1, x_c)] \to \infty$, which together with

$x_c \to b/c > 0$,would imply that the right hand side of (A.49) must diverge to infinity, thus contradicting the constancy of $N_0^c(N_1)$. Hence (A.58) must hold, and the result is established. ■

In particular, the uniqueness of x_c ensures the existence of a well-defined *core-boundary function*, $x_c(N_1)$, satisfying the identity

$$\phi[N_1, x_c(N_1)] = N_1 \tag{A.59}$$

Given this fixed-point property, can now be shown that

Theorem A.2 (Semi-Equilibria). *For each subvector of admissible parameters,* $\widetilde{\theta} = (\rho, \gamma, s_0, \lambda, N, \sigma, \alpha, \beta, b, w, c)$*, and each employment level,* $N_1 \in (N_1^{\min}, N_1^{\max}]$*, there exists a unique semi-equilibrium,* $\widetilde{\xi}(N_1) = (N_0^c, N_0^p, N_1, p_h, U_0^c, U_0^p, U_1, x_c, x_p)$*, for* $\widetilde{\theta}$*.*

Proof. Given N_1, let $x_c = x_c(N_1)$ in (A.59) above. It then suffices to produce a unique set of admissible values $(N_0^c, N_0^p, p_h, U_0^c, U_0^p, U_1, x_p)$ satisfying all conditions of Definition 2. To do so, we begin by taking (N_0^c, N_0^p, p_h) to be the unique steady-state values defined by N_1 in Proposition 3, so that (5.28), (5.29), and (5.30) are automatically satisfied. Using $N_0^c = N_0^c(N_1)$, we then define $U_0^c = U_0^c(N_1)$ by (A.49). Next, defining $r_c = r_c(x_c)$ by (A.46), we set $U_1 = U_1(N_1, x_c)$ in (A.50). Similarly, defining $r_p = r_p(N_1, x_c)$ by (A.47), we set $U_0^p = U_1 r_p$ and set $x_p = x_p(N_1, x_c)$ in (A.48). By construction, these values automatically satisfy [(5.15),(5.18),(5.19),(5.23)]. Finally by (A.52) in Proposition 4 together with the definition of ϕ in (A.51), we see that (5.24) is also satisfied, and hence that these values constitute a semi-equilibrium for $\widetilde{\theta}$. But since x_c in Proposition 4 is unique, it follows from the above argument that all other values $(N_0^c, N_0^p, p_h, U_0^c, U_0^p, U_1, x_p)$ are unique, and hence that this is the only semi-equilibrium for $\widetilde{\theta}$ with employment level N_1. ■

A.3.2. Proof of Theorem 5.1: Uniqueness of *CP* Equilibria

First suppose there are two distinct *CP*-equilibria for θ, say

$$\xi = (N_0^c, N_0^p, N_1, p_h, U_0^c, U_0^p, U_1, x_c, x_p, x_f) \tag{A.60}$$

$$\widetilde{\xi} = (\widetilde{N}_0^c, \widetilde{N}_0^p, \widetilde{N}_1, \widetilde{p}_h, \widetilde{U}_0^c, \widetilde{U}_0^p, \widetilde{U}_1, \widetilde{x}_c, \widetilde{x}_p, \widetilde{x}_f) \tag{A.61}$$

We then claim that $N_1 \neq \widetilde{N}_1$. For if $N_1 = \widetilde{N}_1$, it would follow from the construction of semi-equilibria in the proof of Theorem A.2 above that all components of ξ and $\widetilde{\xi}$ must be the same, except possibly the frontier locations,

x_f and $\widetilde{x}_f$. But the equality $\widetilde{U}_0^p = U_0^p$, together with equilibrium condition (5.33), would then imply that $x_p = \widetilde{x}_p$, and hence that $\xi = \widetilde{\xi}$. Thus we may henceforth assume without loss of generality that

$$\widetilde{N}_1 > N_1 \tag{A.62}$$

In this setting, we first observe from the strict monotonicty properties established in Proposition 3 that the steady-state values $(\widetilde{N}_0^c, \widetilde{N}_0^p, \widetilde{p}_h)$ in $\widetilde{\xi}$ must satisfy the strict inequalities:

$$\widetilde{N}_0^c \;>\; N_0^c \tag{A.63}$$

$$\widetilde{N}_0^p \;<\; N_0^p \tag{A.64}$$

$$\widetilde{p}_h \;<\; p_h \tag{A.65}$$

In addition the last inequality also implies from (5.16) that

$$\tau(\widetilde{p}_h) > \tau(p_h) \tag{A.66}$$

Given these preliminary observations, we first show that both of the following inequalities must hold:

$$\widetilde{x}_p < x_p \tag{A.67}$$

$$\widetilde{U}_1 < U_1 \tag{A.68}$$

To do so, we consider two possible cases: $\widetilde{x}_c \geq x_c$ and $\widetilde{x}_c < x_c$. First suppose that $\widetilde{x}_c \geq x_c$. Then by (A.46) together with $b < w$, it follows that $r_c(\widetilde{x}_c) \leq r_c(x_c)$. But this together with (A.66) is easily seen to imply from (A.47) that

$$\widetilde{r}_p \equiv r_p(\widetilde{N}_1, \widetilde{x}_c) < r_p(N_1, x_c) \equiv r_p \tag{A.69}$$

Moreover, since this in turn implies that $r_p^{\frac{1}{\alpha+\beta}} - \widetilde{r}_p^{\frac{1}{\alpha+\beta}} > 0$, and since (A.48) implies that

$$\begin{aligned} \widetilde{x}_p \quad &= \quad \frac{w \cdot \widetilde{r}_p^{\frac{1}{\alpha+\beta}} - b}{c\left(\widetilde{r}_p^{\frac{1}{\alpha+\beta}} - s_0\right)} < \frac{w \cdot r_p^{\frac{1}{\alpha+\beta}} - b}{c\left(r_p^{\frac{1}{\alpha+\beta}} - s_0\right)} = x_p \\ &\Leftrightarrow \quad \left(w \cdot \widetilde{r}_p^{\frac{1}{\alpha+\beta}} - b\right)\left(r_p^{\frac{1}{\alpha+\beta}} - s_0\right) < \left(w \cdot r_p^{\frac{1}{\alpha+\beta}} - b\right)\left(\widetilde{r}_p^{\frac{1}{\alpha+\beta}} - s_0\right) \\ &\Leftrightarrow \quad w\left(r_p^{\frac{1}{\alpha+\beta}} - \widetilde{r}_p^{\frac{1}{\alpha+\beta}}\right) < \frac{b}{s_0}\left(r_p^{\frac{1}{\alpha+\beta}} - \widetilde{r}_p^{\frac{1}{\alpha+\beta}}\right) \end{aligned} \tag{A.70}$$

it then follows from (4.9) that (A.67) must hold. To establish (A.68), observe next from (A.51),(A.53) and (A.62) that

$$\int_{\widetilde{x}_c}^{\widetilde{x}_p} \eta_1(x, \widetilde{U}_1)dx = \widetilde{N}_1 > N_1 = \int_{x_c}^{x_p} \eta_1(x, U_1)dx \tag{A.71}$$

But this together with $\widetilde{x}_c \geq x_c$ and (A.67) implies that $\eta_1(\cdot, \widetilde{U}_1) > \eta_1(\cdot, U_1)$ on a set of positive measure, and hence from (5.20) that (A.68) must hold.

Next suppose that $\widetilde{x}_c < x_c$. For this case we can establish (A.67) as follows. First recall from part (i) of Proposition 3 that

$$\widetilde{N}_1 > N_1 \Rightarrow \frac{\widetilde{N}_0^c}{\widetilde{N}_1} > \frac{N_0^c}{N_1} \tag{A.72}$$

Hence if we now let

$$F_1(x_c) = \int_0^{x_c} x(b - cx)^{\beta/\alpha} dx \tag{A.73}$$

$$F_2(x_c, x_p) = \int_{x_c}^{x_p} x(w - cx)^{\beta/\alpha} dx \tag{A.74}$$

then by (5.20),(5.21),(5.23), and (5.24), together with (5.18) and (A.46), it follows that

$$\begin{aligned} \frac{\widetilde{N}_0^c}{\widetilde{N}_1} > \frac{N_0^c}{N_1} \quad &\Leftrightarrow \quad \left(\frac{\widetilde{U}_0^c}{\widetilde{U}_1}\right)^{-\frac{1}{\alpha}} \frac{F_1(\widetilde{x}_c)}{F_2(\widetilde{x}_c, \widetilde{x}_p)} > \left(\frac{U_0^c}{U_1}\right)^{-\frac{1}{\alpha}} \frac{F_1(x_c)}{F_2(x_c, x_p)} \\ &\Leftrightarrow \quad r_c(\widetilde{x}_c)^{-\frac{1}{\alpha}} \frac{F_1(\widetilde{x}_c)}{F_2(\widetilde{x}_c, \widetilde{x}_p)} > r_c(x_c)^{-\frac{1}{\alpha}} \frac{F_1(x_c)}{F_2(x_c, x_p)} \end{aligned} \tag{A.75}$$

But since $\widetilde{x}_c < x_c \Rightarrow r_c(\widetilde{x}_c) > r_c(x_c) \Rightarrow r_c(\widetilde{x}_c)^{-\frac{1}{\alpha}} < r_c(x_c)^{-\frac{1}{\alpha}}$, we must have

$$\frac{F_1(\widetilde{x}_c)}{F_2(\widetilde{x}_c, \widetilde{x}_p)} > \frac{F_1(x_c)}{F_2(x_c, x_p)} \tag{A.76}$$

Moreover, since $\widetilde{x}_c < x_c \Rightarrow F_1(\widetilde{x}_c) < F_1(x_c)$ by (A.73), it then follows from (A.76) that

$$F_2(\widetilde{x}_c, \widetilde{x}_p) < F_2(x_c, x_p) \tag{A.77}$$

Finally, since $F_2(x_c, x_p)$ is seen from (A.74) to be decreasing in x_c and increasing in x_p, we may conclude from (A.77) that

$$\begin{aligned} \widetilde{x}_c < x_c \quad &\Rightarrow \quad F_2(x_c, \widetilde{x}_p) < F_2(\widetilde{x}_c, \widetilde{x}_p) < F_2(x_c, x_p) \\ &\Rightarrow \quad \widetilde{x}_p < x_p \end{aligned} \tag{A.78}$$

and hence that (A.67) must hold.

To establish (A.68) for this second case, observe first from (5.18) and (A.46) that

$$\begin{aligned} \widetilde{x}_c < x_c \quad &\Rightarrow \quad r_c(\widetilde{x}_c) > r_c(x_c) \\ &\Rightarrow \quad \frac{\widetilde{U}_0^c}{\widetilde{U}_1} > \frac{U_0^c}{U_1} \end{aligned} \tag{A.79}$$

Moreover, from (5.23) and (A.63) we see that

$$\int_0^{\widetilde{x}_c} \eta_0^c(x, \widetilde{U}_0^c)dx = \widetilde{N}_0^c > N_0^c = \int_0^{x_p} \eta_0^c(x, U_0^c)dx \tag{A.80}$$

which together with $\widetilde{x}_c < x_c$ implies that $\eta_0^c(\cdot, \widetilde{U}_0^c) > \eta_0^c(\cdot, U_0^c)$ on a set of positive measure. Hence from (5.21) it follows that $\widetilde{U}_0^c < U_0^c$, and we may conclude from (A.79) that (A.68) must hold.

Given the two conditions, (A.67) and (A.68), we can now establish the desired uniqueness result by obtaining a contradiction as follows. Observe first from (5.25) and (A.64) that

$$\int_{\widetilde{x}_p}^{\widetilde{x}_f} \eta_0^p(x, \widetilde{U}_0^p)dx = \widetilde{N}_0^p < N_0^p = \int_{x_p}^{x_f} \eta_0^p(x, U_0^p)dx \tag{A.81}$$

But since (5.19) together with (4.9) and (A.67) imply that

$$\begin{aligned} \frac{\widetilde{U}_0^p}{\widetilde{U}_1} &= s_0^{\alpha+\beta}\left(\frac{(b/s_0) - c\widetilde{x}_p}{w - c\widetilde{x}_p}\right)^{\alpha+\beta} \\ &< s_0^{\alpha+\beta}\left(\frac{(b/s_0) - cx_p}{w - cx_p}\right)^{\alpha+\beta} = \frac{U_0^p}{U_1} \end{aligned} \tag{A.82}$$

we see from (A.68) that $\widetilde{U}_0^p < U_0^p$, and hence that $\eta_0^p(x, \widetilde{U}_0^p) > \eta_0^p(x, U_0^p)$ for all x. Hence this together with (A.67) and (A.81) implies on the one hand that $\widetilde{x}_f < x_f$. But on the other hand, $\widetilde{U}_0^p < U_0^p$ and together with equilibrium condition (5.33) implies that $\widetilde{x}_f > x_f$. Thus the assumption that ξ and $\widetilde{\xi}$ are distinct *CP*-equilibria for θ leads to a contradiction, and the uniqueness of *CP*-equilibria is established. ■

A.3.3. Proof of Theorem 5.2: Existence of *CP* Equilibria

To establish conditions for the existence of *CP*-equilibria, we first construct a limiting semi-equilibrium associated with the minimal employment level, $N_1^{\min}$, in (A.31) above. The strategy of the existence proof is then to establish conditions on parameters under which semi-equilibria 'sufficiently close' to this limit will indeed be *CP*-equilibria.

To construct the desired limit, we begin by recalling that each minimal employment level is uniquely defined by the steady-state parameters, $\overline{\theta} = (\rho, \gamma, s_0, \lambda, N)$. To emphasize this dependence on $\overline{\theta}$, we now let $\overline{N}_1\,[= N_1^{\min}(\overline{\theta})]$ denote the unique minimal employment level *generated* by $\overline{\theta}$. Next we show that for any extension of $\overline{\theta}$ to an admissible subvector of parameters, $\widetilde{\theta} =$

$(\overline{\theta}, \sigma, \alpha, \beta, b, w, c)$, this minimal employment level, $\overline{N}_1$, can be extended to a unique vector of values,

$$\overline{\xi} = (\overline{N}_0^c, \overline{N}_0^p, \overline{N}_1, \overline{p}_h, \overline{U}_0^c, \overline{U}_0^p, \overline{U}_1, \overline{x}_c, \overline{x}_p) \tag{A.83}$$

which constitutes a semi-equilibrium for $\widetilde{\theta}$. To start with, observe that since $\overline{N}_1$ is by definition associated with the minimal level of search intensity, s_0, it follows from (5.26) that if $\overline{\xi}$ is a semi-equilibrium, then we must have

$$s_0 = \frac{\overline{N}_0^c + s_0 \overline{N}_0^p}{\overline{N}_0^c + \overline{N}_0^p} \Rightarrow s_0 \overline{N}_0^c = \overline{N}_0^c \Rightarrow \overline{N}_0^c = 0 \tag{A.84}$$

Hence, in terms of core-periphery patterns, $\overline{\xi}$ is seen to represent the limiting case in which the core unemployment level, $\overline{N}_0^c$ just falls to zero. In view of (5.23), it then follows that the core boundary point in $\overline{\xi}$ must be

$$\overline{x}_c = 0 \tag{A.85}$$

To generate the remaining values in $\overline{\xi}$, observe first from (A.84) and (5.29) that the peripheral unemployment level must be

$$\overline{N}_0^p = N - \overline{N}_1 \tag{A.86}$$

and also from (5.29) together with Proposition 3 that the steady-state hiring probability must be

$$\overline{p}_h = p_h(\overline{N}_1) = \frac{s_0 \left(N - \overline{N}_1\right)}{\left(N - \overline{N}_1\right) + Nd} \left(1 - e^{-\gamma \frac{s_0 (N - \overline{N}_1)}{(N - \overline{N}_1) + Nd}}\right) \tag{A.87}$$

By (A.46) and (A.47) together with (5.16), it then follows that the utility ratios, $\overline{r}_c = \overline{U}_0^c / \overline{U}_1$ and $\overline{r}_p = \overline{U}_0^p / \overline{U}_1$ must be given respectively by

$$\overline{r}_c = \left(\frac{b}{w}\right)^{\alpha+\beta} \tag{A.88}$$

$$\overline{r}_p = \tau\left(\overline{p}_h\right) \left(\frac{b}{w}\right)^{\alpha+\beta} + 1 - \tau\left(\overline{p}_h\right) \tag{A.89}$$

In particular, (A.89) implies that the peripheral employment boundary must given in terms of (A.48) by

$$\overline{x}_p = \frac{w \cdot \overline{r}_p^{\frac{1}{\alpha+\beta}} - b}{c\left(\overline{r}_p^{\frac{1}{\alpha+\beta}} - s_0\right)} \tag{A.90}$$

Finally, to determine the utility levels associated with $\overline{\xi}$, it is convenient to begin by employing (5.24) together with (A.90) to determine $\overline{U}_1$ by the relation,

$$\overline{N}_1 = \int_0^{\overline{x}_p} \eta_1(x, \overline{U}_1)dx \tag{A.91}$$

[which is seen from (5.24) to yield a closed-form expression for $\overline{U}_1$]. This in turn allows $\overline{U}_0^c$ and $\overline{U}_0^p$ to be determined respectively by

$$\overline{U}_0^c = \overline{r}_c \cdot \overline{U}_1 \tag{A.92}$$

$$\overline{U}_0^p = \overline{r}_p \cdot \overline{U}_1 \tag{A.93}$$

Given these values, it is then shown that $\overline{\xi}$ is in fact the continuous limit of semi-equilibria, $\widetilde{\xi}(N_1)$, for $\widetilde{\theta} = (\overline{\theta}, \sigma, \alpha, \beta, b, w, c)$, as N_1 approaches $\overline{N}_1$:

Proposition 5. *For each vector of admissible parameters, $\widetilde{\theta} = (\overline{\theta}, \sigma, \alpha, \beta, b, w, c)$, with steady-state parameter vector, $\overline{\theta}$, the state vector, $\overline{\xi}$, is a semi-equilibrium for $\widetilde{\theta}$, and in addition:*

$$\lim_{N_1 \downarrow \overline{N}_1} \widetilde{\xi}(N_1) = \overline{\xi} \tag{A.94}$$

Proof. To establish (A.94), observe first from the continuity of all functions used in the constructions from (A.84) to (A.93) [together with the preservation of equalities under limits] that the result will follow at once if it can be shown that the core-boundary function, $x_c(N_1)$, defined in (A.59) above converges to zero, i.e., that

$$\lim_{N_1 \downarrow \overline{N}_1} x_c(N_1) = 0 \tag{A.95}$$

To do so, suppose to the contrary that (A.95) fails. Then for any sequence of values $(N_1^m) \subseteq (N_1^{\min}, N_1^{\max}]$ with $\lim_{m\to\infty} N_1^m = \overline{N}_1$ it follows from the boundedness of $x_c(N_1) \in [0, b/c]$ (together with the Bolzano-Weierstrass Theorem) that without loss of generality we may assume the existence of a positive limit for the sequence, $x_c^m = x_c(N_1^m)$, say

$$\lim_{m\to\infty} x_c^m = \overline{x}_c \in (0, b/c] \tag{A.96}$$

But the argument in (A.84) shows that

$$\lim_{m\to\infty} N_0^c(N_1^m) = 0 \tag{A.97}$$

which in turn implies from (5.23) together with (A.96) that

$$\lim_{m\to\infty} U_0^c(N_1^m) = \infty \tag{A.98}$$

Moreover, since (A.96) also implies from (A.46) that

$$\lim_{m\to\infty} r_c[x_c(N_1^m)] = \left(\frac{b - c\overline{x}_c}{w - \overline{x}_c}\right)^{\alpha+\beta} \in \left[0, (b/c)^{\alpha+\beta}\right), \tag{A.99}$$

it would then follow from (A.50) that

$$\lim_{m\to\infty} U_1(N_1^m, x_c^m) = \infty \tag{A.100}$$

Finally, since this together with the boundedness of $x_p(N_1^m, x_c^m)$ in (A.48) would imply from (A.51) that

$$\lim_{m\to\infty} \phi(N_1^m, x_c^m) = \lim_{m\to\infty} \int_{x_c^m}^{x_p(N_1^m, x_c^m)} \eta_1[x, U_1(N_1^m, x_c^m)]dx = 0 \tag{A.101}$$

we see from the identity $\phi(N_1^m, x_c^m) \equiv N_1^m$[in (A.52)] that this would contradict the positivity of $\overline{N}_1 = N_1^{\min}(\overline{\theta})$ [as implied by Theorem 1]. Hence failure of (A.95) leads to a contradiction, and we may conclude that (A.95) must hold. ■

In view of the definition of $\overline{N}_1$ as the minimal employment level, $N_1^{\min}(\overline{\theta})$, we now designate $\overline{\xi}$ as the *minimal semi-equilibrium* for $\widetilde{\theta}$. Proposition 5 then shows how this (easily computed) minimal semi-equilibrium can be used to draw inferences about the properties of semi-equilibria, $\widetilde{\xi}(N_1)$, in the neighborhood of $\overline{\xi}$. With this in mind, we first show that the optimality conditions (5.31) and (5.32) are *always* satisfied by the minimal semi-equilibrium, $\overline{\xi}$. To state this result, it is convenient to specialize the general functional definition in (3.9) to the present case. For any parameter vector, $\widetilde{\theta} = (\rho, \gamma, s_0, \lambda, N, \sigma, \alpha, \beta, b, w, c)$, and state vector, $\widetilde{\xi} = (N_0^c, N_0^p, N_1, p_h, U_0^c, U_0^p, U_1, x_c, x_p)$, let $x(1, \widetilde{\theta}, \widetilde{\xi})$ be defined by (3.9) together with (3.5) and (3.6), where $s = 1$ and $U_0 = U_0^c$. Similarly, let $x(s_0, \widetilde{\theta}, \widetilde{\xi})$ be defined by (3.9) together with (3.5) and (3.6), where $s = s_0$ and $U_0 = U_0^p$. With this notation, we have that:

Proposition 6. *For each vector of admissible parameters,* $\widetilde{\theta} = (\overline{\theta}, \sigma, \alpha, \beta, b, w, c)$, *the minimal semi-equilibrium,* $\overline{\xi}$, *for* $\widetilde{\theta}$ *in (A.83) satisfies the following two conditions:*

$$(i) \qquad \overline{x}_c < x(1, \widetilde{\theta}, \overline{\xi}) \tag{A.102}$$

$$(ii) \qquad \overline{x}_p > x(1, \widetilde{\theta}, \overline{\xi}) \tag{A.103}$$

Proof. To establish (i) observe first that by substituting (3.5) and (3.6) into (3.9) [with $s = 1$ and $U_0 = \overline{U}_0^c$] and reducing, we obtain the expression

$$x\left(1, \widetilde{\theta}, \overline{\xi}\right) = \left(\frac{b}{c}\right)\left(\frac{\sigma\overline{p}_h\left(\overline{U}_1 - \overline{U}_0^c\right)}{(\alpha+\beta)(A + \sigma\overline{p}_h)\overline{U}_0^c + \sigma\overline{p}_h\left(\overline{U}_1 - \overline{U}_0^c\right)}\right) \tag{A.104}$$

where $A = 1 - \sigma + \sigma\rho \in (0,1)$. But since $\overline{U}_0^c/\overline{U}_1 = \overline{r}_c = (b/w)^{\alpha+\beta}$ by (A.88) it then follows [dividing through (A.104) by $\overline{U}_0^c$] that

$$x\left(1,\widetilde{\theta},\overline{\xi}\right) = \left(\frac{b}{c}\right)\left(\frac{\sigma\overline{p}_h\left[\left(\frac{w}{b}\right)^{\alpha+\beta} - 1\right]}{(\alpha+\beta)(A+\sigma\overline{p}_h) + \sigma\overline{p}_h\left[\left(\frac{w}{b}\right)^{\alpha+\beta} - 1\right]}\right) \tag{A.105}$$

which is clearly positive since $w > b$. Hence we may conclude from (A.85) that (i) must hold.

To establish (ii) observe that for $s = s_0$ we obtain the following parallel of expression (A.104)

$$x\left(1,\widetilde{\theta},\overline{\xi}\right) = \left(\frac{b}{s_0 c}\right)\left(\frac{s_0\sigma\overline{p}_h\left(\overline{U}_1 - \overline{U}_0^p\right)}{(\alpha+\beta)(A+s_0\sigma\overline{p}_h)\overline{U}_0^p + s_0\sigma\overline{p}_h\left(\overline{U}_1 - \overline{U}_0^p\right)}\right) \tag{A.106}$$

Now observing from (A.89) that $\overline{U}_0^p/\overline{U}_1 = \overline{r}_p = \overline{\tau}z^\mu + 1 - \overline{\tau}$, where $\overline{\tau} = \tau(\overline{p}_h)$, $z = b/w$, and $\mu = \alpha+\beta$, we may divide through (A.106) by $\overline{U}_1$ to obtain

$$x\left(1,\widetilde{\theta},\overline{\xi}\right) = \left(\frac{b}{s_0 c}\right)\left(\frac{s_0\sigma\overline{p}_h\overline{\tau}(1-z^\mu)}{\mu(A+s_0\sigma\overline{p}_h)\left(\overline{\tau}z^\mu + 1 - \overline{\tau}\right) + s_0\sigma\overline{p}_h\overline{\tau}(1-z^\mu)}\right) \tag{A.107}$$

Hence writing $\overline{x}_p$ in (A.90) as

$$\begin{aligned}\overline{x}_p &= \frac{w\cdot\overline{r}_p^{\frac{1}{\alpha+\beta}} - b}{c\left(\overline{r}_p^{\frac{1}{\alpha+\beta}} - s_0\right)} = \left(\frac{b}{s_0 c}\right)\left(\frac{\frac{s_0}{z}\left(\overline{\tau}z^\mu + 1 - \overline{\tau}\right)^{1/\mu} - s_0}{\left(\overline{\tau}z^\mu + 1 - \overline{\tau}\right)^{1/\mu} - s_0}\right) \qquad \text{(A.108)}\\ &= \left(\frac{b}{s_0 c}\right)\left(\frac{\frac{s_0}{z}\left(\overline{\tau}z^\mu + 1 - \overline{\tau}\right)^{1/\mu} - s_0}{\left(1-\frac{s_0}{z}\right)\left(\overline{\tau}z^\mu + 1 - \overline{\tau}\right)^{1/\mu} + \left[\frac{s_0}{z}\left(\overline{\tau}z^\mu + 1 - \overline{\tau}\right)^{1/\mu} - s_0\right]}\right)\end{aligned}$$

and observing that for any positive numbers W, B, C, D

$$\frac{W}{W+B} \geq \frac{C}{C+D} \Leftrightarrow \frac{W}{B} \geq \frac{C}{D} \tag{A.109}$$

it follows from a comparison of (A.107) and (A.108) that (ii) will hold iff

$$\frac{\frac{s_0}{z}\left(\overline{\tau}z^\mu + 1 - \overline{\tau}\right)^{1/\mu} - s_0}{\left(1-\frac{s_0}{z}\right)\left(\overline{\tau}z^\mu + 1 - \overline{\tau}\right)^{1/\mu}} > \frac{s_0\sigma\overline{p}_h\overline{\tau}(1-z^\mu)}{\mu(A+s_0\sigma\overline{p}_h)\left(\overline{\tau}z^\mu + 1 - \overline{\tau}\right)} \tag{A.110}$$

which can be equivalently written as

$$\frac{1}{z-s_0}\left(1 - \frac{z}{\left(\overline{\tau}z^\mu + 1 - \overline{\tau}\right)^{1/\mu}}\right) > \frac{\sigma\overline{p}_h\overline{\tau}}{(A+s_0\sigma\overline{p}_h)\left(\overline{\tau}z^\mu + 1 - \overline{\tau}\right)}\left(\frac{1-z^\mu}{\mu}\right) \tag{A.111}$$

Finally, recalling from (5.16) that

$$\overline{\tau} = \frac{A + s_0\sigma\overline{p}_h}{A + \sigma\overline{p}_h} \;\Rightarrow\; \frac{\sigma\overline{p}_h\overline{\tau}}{A + s_0\sigma\overline{p}_h} = \frac{\sigma\overline{p}_h}{A + \sigma\overline{p}_h} = \frac{1-\overline{\tau}}{1-s_0} \tag{A.112}$$

it follows that (ii) will hold iff

$$1 - \frac{z}{(\overline{\tau}z^\mu + 1 - \overline{\tau})^{1/\mu}} > \frac{z - s_0}{1 - s_0}\left(\frac{1-\overline{\tau}}{\overline{\tau}z^\mu + 1 - \overline{\tau}}\right)\left(\frac{1 - z^\mu}{\mu}\right) \tag{A.113}$$

To analyze this relation, we focus on the parameter, $\mu = \alpha + \beta$, and for notational simplicity we henceforth drop the bar on τ. Letting $g(\mu)$ be defined by $g(\mu) = g_1(\mu) - g_2(\mu)$, where

$$g_1(\mu) = 1 - \frac{z}{(\tau z^\mu + 1 - \tau)^{1/\mu}} \tag{A.114}$$

$$g_2(\mu) = \frac{z - s_0}{1 - s_0}\left(\frac{1-\tau}{\tau z^\mu + 1 - \tau}\right)\left(\frac{1 - z^\mu}{\mu}\right) \tag{A.115}$$

it follows that verifying (A.113) is equivalent to showing that [for each $z = b/w$] the function $g(\cdot)$ is positive on $(0,1)$. To do so, our strategy will be to show that

$$\lim_{\mu \downarrow 0} g(\mu) > 0 \tag{A.116}$$

and that

$$\mu \in (0,1) \Rightarrow g'(\mu) > 0 \tag{A.117}$$

To establish (A.116) we first observe that by an application of L'Hospital's rule to the log of $(\tau z^\mu + 1 - \tau)^{1/\mu}$, one obtains the limiting relation

$$\lim_{\mu \downarrow 0} (\tau z^\mu + 1 - \tau)^{1/\mu} = z^\tau \tag{A.118}$$

and also by an application of L'Hospital that

$$\lim_{\mu \downarrow 0} \left(\frac{1 - z^\mu}{\mu}\right) = -\ln(z) \tag{A.119}$$

Hence it follows that

$$\lim_{\mu \downarrow 0} g(\mu) = \left(1 - z^{1-\tau}\right) - \frac{z - s_0}{1 - s_0}\left(\frac{1-\tau}{1}\right)[-\ln(z)] \tag{A.120}$$

If we express the right hand side as $f(z) = f_1(z) - f_2(z)$ where

$$f_1(z) = 1 - z^{1-\tau} \tag{A.121}$$

$$
\begin{aligned}
f_2(z) &= \frac{z-s_0}{1-s_0}\left(\frac{1-\tau}{1}\right)[-\ln(z)] \\
&= \frac{z-s_0}{1-s_0}\ln\left(z^{\tau-1}\right) \qquad \text{(A.122)}
\end{aligned}
$$

and recall that $s_0 < z = b/w < 1$, then it suffices to show that for each $\tau \in (0,1)$,

$$z \in (s_0, 1) \Rightarrow f(z) > 0 \qquad \text{(A.123)}$$

Here our strategy is to show that $f(1) = 0$, $f'(1) = 0$, and $f''(z) > 0$ for $z \in (s_0, 1)$, so that $f(z)$ must decrease to zero on $(s_0, 1)$. First, it follows by an inspection of (A.121) and (A.122) that $f(1) = 0$. Next observe that

$$f_1'(z) = (\tau-1)z^{-\tau} \Rightarrow f_1'(1) = \tau - 1 \qquad \text{(A.124)}$$

and

$$
\begin{aligned}
f_2'(z) &= \frac{1}{1-s_0}\left[\ln\left(z^{\tau-1}\right) + (\tau-1)\left(1-\frac{s_0}{z}\right)\right] \\
\Rightarrow \quad f_2'(1) &= \frac{1}{1-s_0}\left[(\tau-1)(1-s_0)\right] = \tau - 1 \qquad \text{(A.125)}
\end{aligned}
$$

so that $f'(1) = 0$. Finally, since $\tau \in (s_0, 1)$ implies both that

$$f_1''(z) = (1-\tau)\tau z^{-(1+\tau)} > 0 \qquad \text{(A.126)}$$

and that

$$f_2''(z) = \frac{\tau-1}{1-s_0}\left(z^{-1} + \frac{s_0}{z^2}\right) < 0 \qquad \text{(A.127)}$$

it follows that $f''(z) = f_1''(z) - f_2''(z) > 0$ for $z \in (s_0, 1)$, so that (A.123) [and hence (A.116)] holds.

Next to establish (A.117), it must be verified that $g_1'(\mu) > g_2'(\mu)$ for all $\mu \in (0,1)$, where

$$g_1'(\mu) = \frac{z}{\mu^2\left(\tau z^\mu + 1 - \tau\right)^{1/\mu}}\left\{\frac{\tau z^\mu \ln(z^\mu)}{\tau z^\mu + 1 - \tau} - \ln\left(\tau z^\mu + 1 - \tau\right)\right\} \qquad \text{(A.128)}$$

$$g_2'(\mu) = \frac{(z-s_0)(\tau-1)}{\mu^2(1-s_0)(\tau z^\mu + 1 - \tau)}\left\{\frac{\tau z^\mu \ln(z^\mu)(1-z^\mu)}{\tau z^\mu + 1 - \tau} + (1-z^\mu) + z^\mu \ln\left(z^\mu\right)\right\} \qquad \text{(A.129)}$$

To show that

$$
\begin{aligned}
&\frac{z}{\mu^2\left(\tau z^\mu + 1 - \tau\right)^{1/\mu}}\left\{\frac{\tau z^\mu \ln(z^\mu)}{\tau z^\mu + 1 - \tau} - \ln\left(\tau z^\mu + 1 - \tau\right)\right\} > \qquad \text{(A.130)} \\
&\frac{(z-s_0)(1-\tau)}{\mu^2(1-s_0)(\tau z^\mu + 1 - \tau)}\left\{\frac{\tau z^\mu \ln(z^\mu)(1-z^\mu)}{\tau z^\mu + 1 - \tau} + (1-z^\mu) + z^\mu \ln\left(z^\mu\right)\right\}
\end{aligned}
$$

we first observe that the expression in braces on the left hand side of (A.130) will be positive if for each $z^\mu \in (0,1)$ the function

$$k_1(\tau) = \tau z^\mu \ln(z^\mu) - (\tau z^\mu + 1 - \tau)\ln(\tau z^\mu + 1 - \tau) \tag{A.131}$$

is positive on $(0,1)$. But since $k(1) = 0$ and since $z^\mu \in (0,1)$ implies that

$$k_1'(\tau) = z^\mu \ln(z^\mu) + (1 - z^\mu)\left[\ln(\tau z^\mu + 1 - \tau) - 1\right] < 0$$

for all $\tau \in (0,1)$, it follows that $k_1(\tau)$ is indeed positive on $(0,1)$. Hence if the right hand side of (A.130) is negative, we are finished. On the other hand, if it is nonnegative, then since

$$z, s_0 \in (0,1) \Rightarrow z > \frac{z - s_0}{1 - s_0} \tag{A.132}$$

and since

$$\tau z^\mu + 1 - \tau \in (0,1) \Rightarrow (\tau z^\mu + 1 - \tau)^{-1/\mu} > (\tau z^\mu + 1 - \tau)^{-1} \tag{A.133}$$

it follows by setting $x = z^\mu \in (0,1)$ [and cancelling μ^2 in (A.130)] that it is enough to show that

$$\begin{aligned}\frac{\tau x}{\tau x + 1 - \tau}\ln(x) - \ln(\tau x + 1 - \tau) > \\ (1-\tau)\left\{\frac{\tau x \ln(x)(1-x)}{\tau x + 1 - \tau} + x\ln(x) + (1-x)\right\}\end{aligned} \tag{A.134}$$

But since the right hand side can be rewritten as

$$\begin{aligned}(1-\tau)\left\{\frac{\tau x \ln(x)(1-x)}{\tau x + 1 - \tau} + x\ln(x) + (1-x)\right\} \\ = \frac{x\ln(x)}{\tau x + 1 - \tau} - \frac{\tau x \ln(x)}{\tau x + 1 - \tau} + (1-\tau)(1-x)\end{aligned} \tag{A.135}$$

it follows by substituting (A.135) into (A.134), cancelling terms and multiplying through by $(\tau x + 1 - \tau)$ that it is enough to show that for each $\tau \in (0,1)$ the function

$$\begin{aligned}k_2(x) &= (1-\tau)(1-x)(\tau x + 1 - \tau) + x\ln(x) \\ &\quad - (\tau x + 1 - \tau)\ln(\tau x + 1 - \tau)\end{aligned} \tag{A.136}$$

is positive for all $x \in (0,1)$. But since $k_2(1) = 0$, (A.136) will follow if the derivative

$$\begin{aligned}k_2'(x) &= (1-\tau)\tau(1-x) - (1-\tau)(\tau x + 1 - \tau) \\ &\quad + \ln(x) + 1 - \tau\left[\ln(\tau x + 1 - \tau) + 1\right]\end{aligned} \tag{A.137}$$

is negative on $(0,1)$. Moreover, since $k_2'(1) = 0$, this in turn will follow if the second derivative

$$k_2''(x) = \frac{1}{x(\tau x + 1 - \tau)} \{\tau x + 1 - \tau - 2\tau(1-\tau)x[\tau x + (1-\tau)] - \tau^2 x\} \tag{A.138}$$

is positive on $(0,1)$, which is equivalent to showing that for each $\tau \in (0,1)$ the function

$$k_3(x) = \tau x + 1 - \tau - 2\tau(1-\tau)x[\tau x + (1-\tau)] - \tau^2 x \tag{A.139}$$

is positive on $(0,1)$. But since

$$k_3''(x) = -2\tau(1-\tau)(2\tau) < 0 \tag{A.140}$$

we see that $k_3(\cdot)$ is strictly concave, and hence must achieve its minimum on $[0,1]$ at one of the end points. Finally, since $k_3(0) = 1 - \tau > 0$ and $k_3(1) = (1-\tau)^2 > 0$, we may conclude that $k_3(x)$ is indeed positive on $(0,1)$, and thus that (A.117) must hold. ■

Together with the continuity property in (A.94), this result implies that all semi-equilibria, $\widetilde{\xi}$, sufficiently close to $\overline{\xi}$ will also satisfy these optimality conditions. Hence if $\widetilde{\theta}$ is now extended to a full parameter vector, $\theta = (\widetilde{\theta}, R_A)$ for some given agricultural rent level, R_A, then to ensure that such semi-equilibria, $\widetilde{\xi}$, will actually be *CP*-equilibria, it remains to ensure that *peripheral unemployment condition* (5.25) and the *frontier condition* (5.33) are both satisfied for θ.

Here we require additional restrictions on the relevant range of parameter values. While many such conditions are possible (given the high dimensionality of the parameter space), we choose to focus on two alternative sufficient conditions which are most easily interpreted from an economic viewpoint. To motivate the first of these conditions, recall from the discussion leading to (4.9) that in order for there to exist a peripheral unemployment ring, the unemployment-benefit level, b, cannot be too small. Our first sufficient condition essentially involves a strengthening of (4.9) which asserts that if b is 'sufficiently large', then for an appropriately specified range of exogenous rent levels, R_A, there will exist a unique *CP*-equilibrium for $\theta = (\widetilde{\theta}, R_A)$. Our second sufficient condition asserts that if future utility levels are not too important relative to present utility levels for workers, i.e., if their utility discount rate, σ, is 'sufficiently small', then essentially the same existence result holds. While the proof of sufficiency is rather technical, the basic intuition appears to be the same in each case. Both conditions essentially ensure that there are strong incentives for unemployed workers to choose only a minimal level of search intensity. This is most evident in the case of high unemployment benefits, where unemployment becomes a relatively attractive situation. In the case of utility-discount behavior, it is also clear that if the future is not too important, then there is less incentive for unemployed workers to forego current utility by searching for future work at full intensity.

To state these results in a precise way, we proceed in several steps. The first step, which is really the key result, is to show that if either of these conditions holds for a given parameter vector, $\widetilde{\theta} = (\overline{\theta}, \sigma, \alpha, \beta, b, w, c)$, with minimal semi-equilibrium, $\overline{\xi}$, in (A.83), then there will exist a 'feasible frontier' value in the interval $\left(\overline{x}_p, \frac{b}{s_0 c}\right)$ which allows the peripheral unemployment condition (5.25) to be satisfied by the semi-equilibrium values $\left(\overline{N}_0^p, \overline{U}_0^p, \overline{x}_p\right)$. For convenience, we state this result for each condition separately. First, for the case of unemployment-benefit levels, b, we have that:

Proposition 7. *For any admissible parameter vector, $\widehat{\theta} = (\rho, \gamma, s_0, \lambda, N, \sigma, \alpha, \beta, w, c)$, there exists a unique smallest unemployment-benefit level, $\widehat{b} \in (s_0 w, w)$, such that for each $b \in (\widehat{b}, w)$, the minimal semi-equilibrium, $\overline{\xi}$, for $\left(\widehat{\theta}, b\right)$ satisfies the strict inequality*

$$\overline{N}_0^p < \int_{\overline{x}_p}^{\frac{b}{s_0 c}} \eta_0^p(x, \overline{U}_0^p) dx \tag{A.141}$$

Proof. To establish the existence of such an unemployment-benefit value, $\widehat{b} \in (s_0 w, w)$, we begin by letting the minimal semi-equilibrium for each $(\widehat{\theta}, b)$ with $b \in (s_0 w, w)$ be denoted by

$$\overline{\xi}(b) = \left(\overline{N}_0^c, \overline{N}_0^p, \overline{N}_1, \overline{p}_h, \overline{U}_0^c(b), \overline{U}_0^p(b), \overline{U}_1(b), \overline{x}_c, \overline{x}_p(b)\right) \tag{A.142}$$

where the construction of (A.83) shows that the only quantities depending on b are those indicated. For this family of minimal semi-equilibria, observe first from (A.89) that

$$\lim_{b \uparrow w} \overline{r}_p(b) = \lim_{b \uparrow w} \left[\tau(\overline{p}_h) \left(\frac{b}{w}\right)^{\alpha+\beta} + 1 - \tau(\overline{p}_h)\right] = 1 \tag{A.143}$$

which in turn implies from (A.90) that

$$\lim_{b \uparrow w} \overline{x}_p(b) = \lim_{b \uparrow w} \left[\frac{w\left[\overline{r}_p(b)\right]^{\frac{1}{\alpha+\beta}} - b}{c\left(\left[\overline{r}_p(b)\right]^{\frac{1}{\alpha+\beta}} - s_0\right)}\right] = 0 \tag{A.144}$$

Hence if we consider the function defined for each $b \in (s_0 w, w)$ by

$$G(b) = [\overline{r}_p(b)]^{-\frac{1}{\alpha}} \frac{\int_{\overline{x}_p(b)}^{\frac{b}{s_0 c}} x(b - s_0 cx)^{\frac{\beta}{\alpha}} dx}{\int_0^{\overline{x}_p(b)} x(w - cx)^{\frac{\beta}{\alpha}} dx} \tag{A.145}$$

then it follows at once from (A.142), (A.143), and (A.144) that

$$\lim_{b \uparrow w} G(b) = \infty \tag{A.146}$$

But since $\overline{r}_p(b) = \overline{U}_0^p(b)/\overline{U}_1(b)$ by (A.93), it also follows from (5.20) and (5.22) that for each $b \in (s_0 w, w)$,

$$\begin{aligned} G(b) &= \left(\frac{\overline{U}_0^p(b)}{\overline{U}_1(b)} \right)^{-\frac{1}{\alpha}} \frac{\int_{\overline{x}_p(b)}^{\frac{b}{s_0 c}} x(b - s_0 cx)^{\frac{\beta}{\alpha}} dx}{\int_0^{\overline{x}_p(b)} x(w - cx)^{\frac{\beta}{\alpha}} dx} \\ &= \frac{2\pi \left(\frac{\alpha+\beta}{\alpha}\right) \left(\frac{a}{\overline{U}_0^p(b)}\right)^{\frac{1}{\alpha}} \int_{\overline{x}_p(b)}^{\frac{b}{s_0 c}} x(b - s_0 cx)^{\frac{\beta}{\alpha}} dx}{2\pi \left(\frac{\alpha+\beta}{\alpha}\right) \left(\frac{a}{\overline{U}_1(b)}\right)^{\frac{1}{\alpha}} \int_0^{\overline{x}_p(b)} x(w - cx)^{\frac{\beta}{\alpha}} dx} \\ &= \frac{\int_{\overline{x}_p(b)}^{\frac{b}{s_0 c}} \eta_0^p[x, \overline{U}_0^p(b)] dx}{\int_0^{\overline{x}_p(b)} \eta_1[x, \overline{U}_1(b)] dx} \end{aligned} \tag{A.147}$$

Moreover, since (A.91) together with (5.20) implies that

$$\overline{N}_1 = \int_0^{\overline{x}_p(b)} \eta_1[x, \overline{U}_1(b)] dx \tag{A.148}$$

it then follows from (A.146), (A.147), and (A.148) that

$$\lim_{b \uparrow w} \int_{\overline{x}_p(b)}^{\frac{b}{s_0 c}} \eta_0^p[x, \overline{U}_0^p(b)] dx = \lim_{b \uparrow w} \overline{N}_1 G(b) = \infty \tag{A.149}$$

Hence for all sufficiently large $b \in (s_0 w, w)$ we must have

$$\overline{N}_0^p < \int_{\overline{x}_p(b)}^{\frac{b}{s_0 c}} \eta_0^p[x, \overline{U}_0^p(b)] dx \tag{A.150}$$

and the desired result follows by setting $\widehat{b}$ equal to the *infimum* of all $b \in (s_0 w, w)$ satisfying (A.150). ■

Condition (A.141) together with the positivity of the density $\eta_0^p(\cdot, \overline{U}_0^p)$ is then seen to imply that for each parameter vector, $(\widehat{\theta}, b)$, in Proposition 7 there exists a unique *feasible frontier*, $\overline{x}_f^b \in \left(\overline{x}_p, \frac{b}{s_0 c}\right)$, such that the peripheral unemployment condition is satisfied, i.e., such that

$$\overline{N}_0^p = \int_{\overline{x}_p}^{\overline{x}_f^b} \eta_0^p(x, \overline{U}_0^p) dx \tag{A.151}$$

Similarly, for the case of utility-discount rates, σ, we have that:

Proposition 8. *For any admissible parameter vector,* $\widehat{\theta} = (\rho, \gamma, s_0, \lambda, N, \alpha, \beta, b, w, c)$*, there exists a unique largest utility-discount rate,* $\widehat{\sigma} \in (0,1)$*, such that for each* $\sigma \in (0, \widehat{\sigma})$*, the minimal semi-equilibrium,* $\overline{\xi}$*, for* $(\widehat{\theta}, \sigma)$ *satisfies* (A.141).

Proof. The proof of Proposition 8 closely follows that of Proposition 7. To establish the existence of such a utility-discount rate, $\widehat{\sigma} \in (0,1)$, we begin letting the minimal semi-equilibrium for each $(\widehat{\theta}, \sigma)$ with $\sigma \in (0,1)$ be denoted by

$$\overline{\xi}(\sigma) = \left(\overline{N}_0^c, \overline{N}_0^p, \overline{N}_1, \overline{p}_h, \overline{U}_0^c(\sigma), \overline{U}_0^p(\sigma), \overline{U}_1(\sigma), \overline{x}_c, \overline{x}_p(\sigma)\right) \tag{A.152}$$

where again the construction of (A.83) shows that the only quantities depending on σ are those indicated. Then by letting $\tau(\sigma) = \tau(\overline{p}_h, \sigma)$ in (5.16) it follows that

$$\lim_{\sigma \downarrow 0} \tau(\sigma) = \lim_{\sigma \downarrow 0} \frac{(1-\sigma+\sigma\rho) + s_0\sigma\overline{p}_h}{(1-\sigma+\sigma\rho) + s_0\sigma\overline{p}_h} = 1 \tag{A.153}$$

This in turn implies from (A.89) that

$$\lim_{\sigma \downarrow 0} \overline{r}_p(\sigma) = \lim_{\sigma \downarrow 0} \left[\tau(\sigma)\left(\frac{b}{w}\right)^{\alpha+\beta} + 1 - \tau(\sigma)\right] = \left(\frac{b}{w}\right)^{\alpha+\beta} \tag{A.154}$$

and hence from (A.90) together with (4.9) that

$$\lim_{\sigma \downarrow 0} \overline{x}_p(\sigma) = \lim_{\sigma \downarrow 0} \left[\frac{w\,[\overline{r}_p(\sigma)]^{\frac{1}{\alpha+\beta}} - b}{c\left([\overline{r}_p(\sigma)]^{\frac{1}{\alpha+\beta}} - s_0\right)}\right] = \frac{w\left(\frac{b}{w}\right) - b}{c\left(\frac{b}{w} - s_0\right)} = 0 \tag{A.155}$$

Thus, letting the function H be defined for all $\sigma \in (0,1)$ by

$$H(\sigma) = [\overline{r}_p(\sigma)]^{-\frac{1}{\alpha}} \frac{\int_{\overline{x}_p(\sigma)}^{\frac{b}{s_0 c}} x(b - s_0 c x)^{\frac{\beta}{\alpha}} dx}{\int_0^{\overline{x}_p(\sigma)} x(w - cx)^{\frac{\beta}{\alpha}} dx} \tag{A.156}$$

it follows from (A.154) and (A.155) that

$$\lim_{\sigma \downarrow 0} H(\sigma) = \infty \tag{A.157}$$

The rest of the argument is essentially identical to that of Proposition 7 with $H(\sigma)$ replacing $G(b)$. Here, $\widehat{\sigma}$ is taken to be the *supremum* of values $\sigma \in (0,1)$ satisfying (A.150) with b replaced by σ. ■

As in Proposition 7 above, this implies that for each parameter vector, $(\widehat{\theta}, \sigma)$, in Proposition 8 there exists a unique *feasible frontier*, $\overline{x}_f^\sigma \in \left(\overline{x}_p, \frac{b}{s_0 c}\right)$, such that

$$\overline{N}_0^p = \int_{\overline{x}_p}^{\overline{x}_f^\sigma} \eta_0^p(x, \overline{U}_0^p) dx \tag{A.158}$$

Since all subsequent development for both of these cases closely parallel one another, we shall focus on the unemployment-benefit case, and shall mention the utility-discount case only when appropriate.

Next observe that if the feasible frontier, $\overline{x}_f^b$, is added to the minimal semi-equilibrium, $\overline{\xi}$, in (A.94), then the above results show that this state vector

$$\overline{\xi}\left(\widehat{\theta}, b\right) = (\overline{N}_0^c, \overline{N}_0^p, \overline{N}_1, \overline{p}_h, \overline{U}_0^c, \overline{U}_0^p, \overline{U}_1, \overline{x}_c, \overline{x}_p, \overline{x}_f^b) \tag{A.159}$$

satisfies all conditions for a *CP*-equilibrium except the frontier condition (5.33). But since this condition involves the agricultural rent, R_A, which is yet to be specified, it follows that this condition is automatically satisfied for the rent level, $\overline{R}_A$, defined by

$$\overline{R}_A = \left(\frac{a}{\overline{U}_0^p}\right)^{\frac{1}{\alpha}} \left(b - s_0 c \overline{x}_f^b\right)^{\frac{\alpha+\beta}{\alpha}} > 0 \tag{A.160}$$

Hence for the parameter vector, $\widetilde{\theta} = \left(\widehat{\theta}, b\right)$, in Proposition 7 it follows that $\overline{\xi}\left(\widehat{\theta}, b\right)$ satisfies all *CP*-equilibrium conditions for $\left(\widehat{\theta}, b, \overline{R}_A\right)$. Note that since $\overline{N}_0^c = 0$, this equilibrium does not yield a genuine core-periphery pattern of unemployment. But it is clearly a limiting case of such equilibria which, in a manner paralleling the designation of minimal semi-equilibria, can be designated as the *minimal equilibrium* for parameter vector $\widetilde{\theta}$. Moreover, it is equally clear from the continuity result in Proposition 5 together with the *strict* inequalities in (A.102), (A.103), and (A.141), that each semi-equilibrium,

$$\widetilde{\xi}(N_1) = (N_0^c, N_0^p, N_1, p_h, U_0^c, U_0^p, U_1, x_c, x_p) \tag{A.161}$$

for $\widetilde{\theta}$ with N_1 'sufficiently close' to $\overline{N}_1$, will still satisfy these conditions, i.e. that

$$x_c < x\left(1, \widetilde{\theta}, \widetilde{\xi}(N_1)\right) \tag{A.162}$$

$$x_p < x\left(s_0, \widetilde{\theta}, \widetilde{\xi}(N_1)\right) \tag{A.163}$$

$$N_0^p < \int_{x_p}^{\frac{b}{s_0 c}} \eta_0^p(x, U_0^p)dx \tag{A.164}$$

Hence if for each such $\widetilde{\xi}(N_1)$ we now define the appropriate frontier value, $x_f(N_1) \in \left(x_p, \frac{b}{s_0 c}\right)$, [in a manner paralleling (A.151)] to be the unique boundary value satisfying

$$N_0^p = \int_{x_p}^{x_f(N_1)} \eta_0^p(x, U_0^p)dx \tag{A.165}$$

and if we again solve for the appropriate agricultural rent level, $R_A(N_1)$, by

$$R_A(N_1) = \left(\frac{a}{U_0^p}\right)^{\frac{1}{\alpha}} (b - s_0 c x_f(N_1))^{\frac{\alpha+\beta}{\alpha}} \tag{A.166}$$

then it follows at once that the state vector

$$\widetilde{\xi}[N_1] = [\widetilde{\xi}(N_1), x_f(N_1)] \tag{A.167}$$

must be the unique *CP*-equilibrium for the parameter vector $\left(\widetilde{\theta}, R_A(N_1)\right)$. Moreover, if we let $\overline{\overline{N}}_1$ denote the *supremum* of all employment levels, N_1, satisfying these conditions, i.e.,

$$\overline{\overline{N}}_1 = \sup\left\{N_1 > \overline{N}_1 : \widetilde{\xi}(N_1) \textit{satisfies } \text{(A.162),(A.163),(A.164)}\right\} \tag{A.168}$$

[which necessarily exists and is less than $N_1^{\max}(\widetilde{\theta})$] then it also follows that the interval of possible *CP*-equilibrium employment levels for $\widetilde{\theta}$ is given by $\left(\overline{N}_1, \overline{\overline{N}}_1\right)$. If we define the associated limiting frontier value, $\overline{\overline{x}}_f^b$, by

$$\overline{\overline{x}}_f^b = \lim_{N_1 \uparrow \overline{\overline{N}}_1} x_f(N_1) \tag{A.169}$$

and adjoin $\overline{\overline{x}}_f$ to the semi-equilibrium, $\widetilde{\xi}(\overline{\overline{N}}_1)$, for $\overline{\overline{N}}_1$, then as a parallel to $\overline{\xi}(\widehat{\theta}, b)$, it is natural to designate this combined state vector

$$\overline{\overline{\xi}}\left(\widehat{\theta}, b\right) = (\overline{\overline{N}}_0^c, \overline{\overline{N}}_0^p, \overline{\overline{N}}_1, \overline{\overline{p}}_h, \overline{\overline{U}}_0^c, \overline{\overline{U}}_0^p, \overline{\overline{U}}_1, \overline{\overline{x}}_c, \overline{\overline{x}}_p, \overline{\overline{x}}_f^b) \tag{A.170}$$

as the *maximal equilibrium* for $\widetilde{\theta} = (\widehat{\theta}, b)$. Finally, since $R_A(N_1)$ is easily seen to be an increasing function of N_1,[23] it follows that the relevant interval of possible agricultural rents, R_A, generated by these *CP*-equilibria for $\widetilde{\theta}$ is given by $\left(R_A(\overline{N}_1), R_A(\overline{\overline{N}}_1)\right)$, which can be equivalently written in terms of (A.166) as

$$\left(\frac{a}{\overline{U}_0^p}\right)^{\frac{1}{\alpha}} \left(b - s_0 c \overline{x}_f^b\right)^{\frac{\alpha+\beta}{\alpha}} < R_A < \left(\frac{a}{\overline{\overline{U}}_0^p}\right)^{\frac{1}{\alpha}} \left(b - s_0 c \overline{\overline{x}}_f^b\right)^{\frac{\alpha+\beta}{\alpha}} \tag{A.171}$$

This establishes the proof of Theorem 5.2.

[23] To see this observe first from part (*ii*) of Proposition 3 that N_0^p is decreasing in N_1. Moreover, it is shown in (A.67), (A.68), and (A.82) of the Appendix that x_p, U_1, and U_0^p are also decreasing in N_1. But these monotonicty results are seen at once from (5.25) to imply that the frontier value, x_f, must be decreasing as well. Finally, since this together with the decreasing monotonicity of U_0^p implies that the right hand side of (5.33) must increase, it follows that $R_A(N_1)$ in (A.158) must increase with N_1.

As already discussed above, to prove Theorem 5.2, we used a sufficient condition expressed in terms of b (b shoud be sufficiently large). We could also express this sufficient condition in terms of σ. Indeed, in a similar manner, one may easily define minimal and maximal equilibria, $\overline{\xi}(\widehat{\theta},\sigma)$ and $\overline{\overline{\xi}}(\widehat{\theta},\sigma)$, for the parameter vector $(\widehat{\theta},\sigma)$ in Proposition 8. It follows that, if the relevant frontier values in each of these equilibria are denoted respectively by $\overline{x}_f^{\sigma}$ and $\overline{\overline{x}}_f^{\sigma}$, then the relevant range of agricultural rents in this case is given by

$$\left(\frac{a}{\overline{U}_0^p}\right)^{\frac{1}{\alpha}}\left(b - s_0 c\overline{x}_f^{\sigma}\right)^{\frac{\alpha+\beta}{\alpha}} < R_A < \left(\frac{a}{\overline{\overline{U}}_0^p}\right)^{\frac{1}{\alpha}}\left(b - s_0 c\overline{\overline{x}}_f^{\sigma}\right)^{\frac{\alpha+\beta}{\alpha}} \tag{A.172}$$

Hence we have the following parallel result for the utility-discount case:

Theorem A.3 (Existence of *CP*-Equilibria). *For any admissible parameters,* $\widehat{\theta} = (\rho,\gamma,s_0,\lambda,N,\alpha,\beta,b,w,c)$, *and any* $\sigma \in (0,\widehat{\sigma})$ *in Proposition 8 with associated minimal and maximal equilibria,* $\overline{\xi}(\widehat{\theta},\sigma)$, $\overline{\overline{\xi}}(\widehat{\theta},\sigma)$, *there exists for each agricultural rent level,* R_A, *in the interval* (A.172) *a unique CP-equilibrium for* $\theta = (\widehat{\theta},\sigma,R_A)$.

Figure 1: Bid Rent for the Unemployed

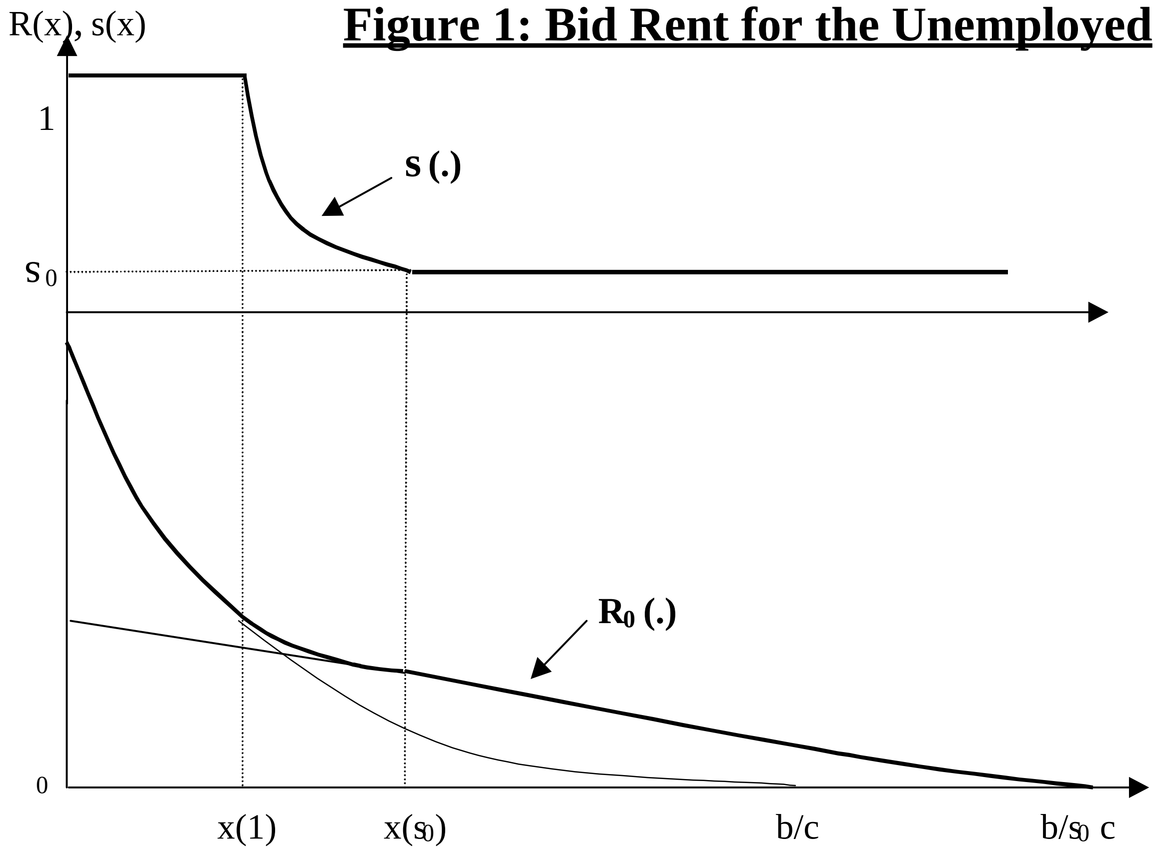
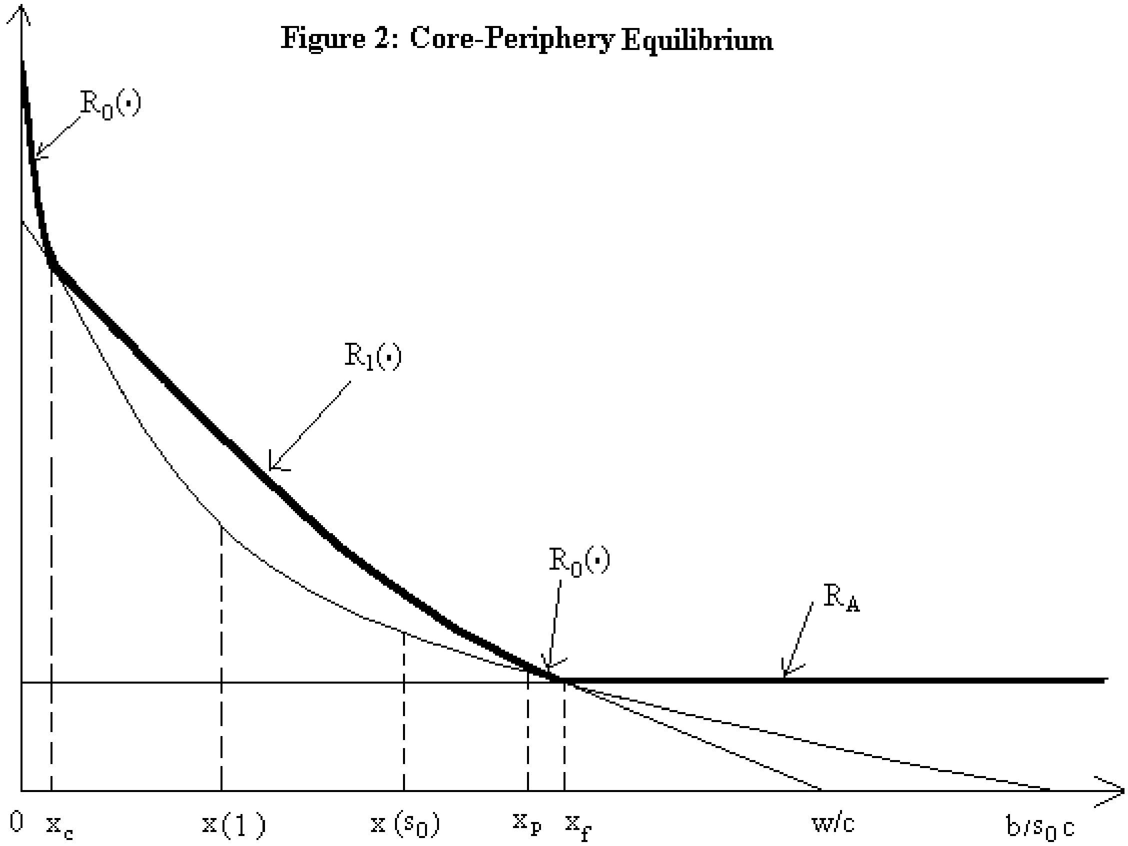


Figure 2: Core-Periphery Equilibrium



IZA Discussion Papers

No.	Author(s)	Title	Area	Date
677	F. Galindo-Rueda A. Vignoles	Class Ridden or Meritocratic? An Economic Analysis of Recent Changes in Britain	2	12/02
678	M. Barbie M. Hagedorn A. Kaul	Fostering Within-Family Human Capital Investment: An Intragenerational Insurance Perspective of Social Security	3	12/02
679	A. Kölling C. Schnabel J. Wagner	Establishment Age and Wages: Evidence from German Linked Employer-Employee Data	1	12/02
680	M. Hagedorn A. Kaul	Langzeitarbeitslosigkeit in Deutschland: Fakten, Ursachen und Bekämpfung	7	12/02
681	M. Hagedorn A. Kaul T. Mennel	An Adverse Selection Model of Optimal Unemployment Insurance	7	12/02
682	B. Cockx	Vocational Training of Unemployed Workers in Belgium	2	01/03
683	E. Strobl	Is Education Used as a Signaling Device for Productivity in Developing Countries? Evidence from Ghana	4	01/03
684	P. A. Puhani	The Rise and Fall of Swiss Unemployment: Relative Demand Shocks, Wage Rigidities, and Temporary Immigrants	2	01/03
685	L. Borghans B. ter Weel	Do We Need Computer Skills to Use a Computer? Evidence from Britain	2	01/03
686	A. S. Kalwij R. Alessie	Permanent and Transitory Wage Inequality of British Men, 1975-2001: Year, Age and Cohort Effects	2	01/03
687	I. Haouas M. Yagoubi A. Heshmati	Labour-Use Efficiency in Tunisian Manufacturing Industries	4	01/03
688	I. Haouas M. Yagoubi A. Heshmati	The Impacts of Trade Liberalization on Employment and Wages in Tunisian Industries	4	01/03
689	A. Heshmati I. Haouas	The Effects of Union Wage-Settings on Firms' Production Factor Decisions	5	01/03
690	M. Hagedorn	Contracting with Private Information	1	01/03
691	B. Petrongolo C. A. Pissarides	Scale Effects in Markets with Search	3	01/03
692	T. E. Smith Y. Zenou	Spatial Mismatch, Search Effort and Urban Spatial Structure	3	01/03

An updated list of IZA Discussion Papers is available on the center‘s homepage www.iza.org.