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ABSTRACT

A Note on Allen-Uzawa Partial Elasticities of Substitution: The Case of the Translog Cost Function

This note provides a useful property of the Allen-Uzawa partials for the translog cost function. It also suggests how the main results extend to any functional form with certain properties. The curvature of the Allen-Uzawa matrix is the same as the curvature of the Hessian matrix. Intuitively and empirically, the Allen-Uzawa partials allow for the verification of curvature properties.

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I. Introduction

Since its inception the Allen-Uzawa partial elasticity of substitution has been a controversial concept.¹ After the seminal work by Uzawa (1962), it has been a widely reported statistic in empirical studies of production. However, Blackorby and Russell (1989) show that the Allen-Uzawa partials (σ_{ij}) are “(incrementally) completely uninformative.” They provide counterfactual examples and conclude that these elasticities are not a measure of the “ease of change” or substitution, reveal no information about relative factor shares, and cannot be interpreted as a logarithmic derivative of quantity ratios to marginal rates of substitution (Blackorby and Russell, 1989, p. 883). They reason the Morishima elasticity of substitution is the natural generalization of the original Hicksian concept. Extending their work, Anderson and Moroney (1993) confirm this to be the case and show how it applies to nested (multi-stage) production technologies. This note, however, deviates from this controversy and addresses a technical property overlooked earlier in the empirical literature.

It shows an unambiguous result for the Allen-Uzawa partials in the special case of the translog (TL) functional form, where its application is straightforward, and extends it to any functional form with certain properties. Its use is somewhat different from what was originally conceived for σ_{ij} . It turns out that the Allen-Uzawa partial elasticity of substitution provides useful information about the curvature of the Hessian matrix in this particular instance, i.e., checking the curvature of the Allen-Uzawa partials matrix (Σ) is the same as checking the curvature of the Hessian matrix of second order partial derivatives of the TL cost

¹ For the original definition of the elasticity of substitution, see Hicks (1932), p. 177 and Allen (1938), p. 340-3, p. 503-9. For an early review of the concept and its uses, refer to Morrisset (1953) and the citations therein.

function with respect to prices. Section two of this note derives the main result. We will also show how the main result of this note extends to any functional form.

Economic theory implies that well-behaved cost functions must be concave with respect to prices. Often, estimated TL cost functions fail to satisfy this property. Section three discusses the issue and shows the implications for Σ when concavity is imposed “globally” as in Diewert and Wales (1987) and “locally” as in Ryan and Wales (2000). The final Section concludes. The appendix offers a step-by-step proof of the main result.

II. The Translog (TL) Cost Function, Concavity and the Allen-Uzawa Partial Elasticities of Substitution (Σ): A Simple Algebra

Following Diewert and Wales (1987), we can specify the TL cost function with n inputs as:

$$\begin{aligned} \ln C(w, y, t) = & b_0 + \sum_{i=1}^n b_i \ln w_i + b_y \ln y + b_t(t - t^*) + (1/2) \sum_{i=1}^n \sum_{j=1}^n b_{ij} \ln w_i \ln w_j \\ & + \sum_{i=1}^n b_{iy} \ln w_i \ln y + \sum_{i=1}^n b_{it}(t - t^*) \ln w_i + (1/2) b_{yy} \ln y \ln y \\ & + b_{yt}(t - t^*) \ln y + (1/2) b_{tt}(t - t^*)^2, \quad b_{ij} = b_{ji} \quad \forall i, j, \end{aligned} \quad (1)$$

where W is a vector of n input prices, y is output, $C(w,y,t)$ is cost, t is time in the sample and t^* is a chosen reference point.² Standard (necessary and sufficient) conditions to ensure that C is linearly homogeneous in w are given by:

² In defining the cost function, we use $(t-t^*)$ instead of t . This facilitates the imposition of local concavity in what follows (see, Ryan and Wales, 2000). They also show that this has no effect on the likelihood function, in estimation, or the elasticities of interest. For regularity conditions on $C(w,y,t)$ see Diewert and Wales (1987, p. 45).

$$\sum_i b_i = 1, \quad \sum_i b_{iy} = 0, \quad \sum_i b_{it} = 0, \quad \text{and} \quad \sum_j b_{ij} = 0 \quad \text{for } i = 1, \dots, n.$$

The share equations for the TL cost function are:

$$S_i(w, y, t) = b_i + \sum_{j=1}^n b_{ij} \ln w_j + b_{iy} \ln y + b_{it}(t - t^*), \quad \text{for } i = 1 \dots n, \quad (2)$$

where $S_i(w, y, t)$ is the i^{th} input's share in the cost function. Assume we set all prices and output to one at the reference point t^* , this implies that $b_i = S_i$ for all i at this point.³ For estimation one uses a system that is completely identified omitting a share equation. Next we derive the main result.

It is known that the logarithmic second order derivatives of a cost function are related to its ordinary first and second order partial derivatives in the following way (see Diewert and Wales, 1987, p. 47):

$$\frac{\partial^2 \ln C(w, y, t)}{\partial \ln w_i \partial \ln w_j} = \frac{\delta_{ij} w_i C_i}{C} - \frac{w_i w_j C_i C_j}{C^2} + \frac{w_i w_j C_{ij}}{C}, \quad (3)$$

where δ_{ij} is the Kronecker delta defined as:

$$\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j, \end{cases} \quad (4)$$

and $C \equiv C(w, y, t)$, $C_i \equiv \partial C(w, y, t) / \partial w_i$, and $C_{ij} \equiv \partial^2 C(w, y, t) / \partial w_i \partial w_j$. For the TL cost function defined in (1), the left-hand side of (3) is the parameter b_{ij} . Let $B \equiv [b_{ij}]$ be a N by N symmetric matrix. Shephard's Lemma gives $C_i = x_i(w, y, t)$ —input demands.

³ Because of duality, all derivations that follow can easily be shown to hold for the indirect translog profit function, see Diewert (1974), Hertel (1984), Diewert and Wales (1987). The derivation of the price elasticities (own and cross) used below is found therein as well.

Define the i^{th} share function (2) in the following way: $S_i(w, y, t) = \frac{w_i x_i(w, y, t)}{C(w, y, t)}$, and

the share vector $S \equiv [S_1 \dots S_n]'$. Let \hat{S} be a N by N *diagonal* matrix with the share vector S on the main diagonal. Define \hat{W} in a similar fashion.⁴ These definitions and a simple algebraic manipulation of (3) result in:

$$\frac{w_i w_j C_{ij}(w, y, t)}{C(w, y, t)} = b_{ij} + S_i S_j - S_i \delta_{ij}. \quad (5)$$

Necessary and sufficient conditions for the concavity property to hold require that the gradient $\nabla^2_{ww} C(w, y, t)$ (in our case C_{ij}) be negative semi-definite for all $w \geq 0_N$, $y > 0$ and $t = 1 \dots T$. (5) can further be expressed as:

$$\frac{\hat{W} [\nabla^2_{ww} C(w, y, t)] \hat{W}}{C(w, y, t)} = B + SS' - \hat{S}, \quad (6)$$

or equivalently:

$$\frac{\hat{W} H \hat{W}}{C} = \Gamma = B + SS' - \hat{S}, \quad (6')$$

where H is the Hessian matrix. This expression gives rise to a useful procedure to impose concavity “globally” as in Diewert and Wales (1987) and “locally” as in Ryan and Wales (1998, 2000).

⁴ More formally, a *diagonal* matrix is defined as $\hat{S} = \left\| S_i \delta_{ij} \right\|$, where S_i varies with $i=1 \dots n$; and $\hat{W} = \left\| w_i \delta_{ij} \right\|$.

Next we express Allen-Uzawa partial elasticity of substitution in terms of price elasticities and shares. For the translog the i^{th} own price elasticity is

$$\eta_{ii} = -1 + S_i + \frac{b_{ii}}{S_i} \text{ and the cross price elasticity } \eta_{ij} = S_j + \frac{b_{ij}}{S_i} \text{ (refer to footnote two).}$$

Given $\sigma_{ij} \equiv \frac{\eta_{ij}}{S_j}$ (see Blackorby and Russell, 1989, p. 883) and since

$$\sigma_{ij} = \frac{b_{ij}}{S_i S_j} - \frac{\delta_{ij}}{S_j} + 1^5, \text{ it is straightforward to show that the parameters } b_{ij} \text{ of the model}$$

have the following representation:

$$b_{ij} = S_i S_j \sigma_{ij} - S_i S_j + S_i \delta_{ij}. \quad (7)$$

In matrix form then (7) becomes

$$B = \hat{S} \Sigma \hat{S} - S S' + \hat{S} \quad (8)$$

where \hat{S} is as above (a diagonal N by N matrix with the share vector S on the main diagonal) and $\Sigma \equiv [\sigma_{ij}]$. A close examination of (6') and (8) reveals the following expression:

$$\Gamma = \hat{S} \Sigma \hat{S} \quad (9)$$

This simply implies, in the case of the TL functional form, that checking the curvature of the matrix of Allen-Uzawa partials is the same as checking the curvature

⁵ For $i = j$, we have $\frac{\eta_{ii}}{S_i} = \sigma_{ii} = -\frac{1}{S_i} + 1 + \frac{b_{ii}}{S_i S_i}$, and for $i \neq j$, $\frac{\eta_{ij}}{S_j} = \sigma_{ij} = 1 + \frac{b_{ij}}{S_i S_j}$. See also

Appendix 1.

of the Hessian matrix. The conclusion is as follows: assuming $C(w,y,t) > 0$ in (6), the Hessian matrix will be negative semi-definite if and only if $B - [\hat{S} - SS']$ is a negative semi-definite matrix. It can be verified that $-\left[\hat{S} - SS'\right]$ is negative semi-definite, provided vector S is nonnegative (i.e., $S \geq 0_N$, refer to Appendix 2). Thus, if B is negative semi-definite, then global concavity of the TL cost function is assured (nonnegative sums of concave functions).⁶ However, (9) suggests

that if Σ is negative semi-definite, it follows the Hessian must also have the same property, provided nonnegative shares and cost. Furthermore, (9) can be expressed

to yield: $\frac{\hat{W}H\hat{W}}{C} = \hat{S}\Sigma\hat{S}$, which rearranging and observing that $\hat{S}^{-1}\hat{W} = C\hat{C}$ ⁷ becomes:

$$C\hat{C}H\hat{C} = \Sigma . \quad (10)$$

As above, \hat{C} is a diagonal matrix with a typical term $1/C_i$. This result provides a simple but useful property of the Allen-Uzawa partials in the case of the TL functional form. The matrix of Allen-Uzawa partials Σ can be used to study and check curvature for this particular functional form. In addition, this result applies for any functional form. As in Uzawa (1962, p. 293), write his equation (9) as

$\sigma_{ij} = \frac{C_{ij}C}{C_iC_j}$, then in matrix form $\Sigma = C\hat{C}H\hat{C}$, where the matrices are as defined

above.

⁶ Diewert and Wales (1987) provide some additional insight. They refer to the work of Diewert, McFadden and Barnett.

⁷ Since $S_i(w, y, t) = \frac{w_i x_i(w, y, t)}{C(w, y, t)} = \frac{w_i C_i}{C}$, and rearranging, in matrix form it gives the desired expression.

III. An Exercise with Curvature Imposed

A. Globally imposed curvature

The TL cost function will satisfy the concavity in prices property globally if the matrix B is negative semi-definite. A general procedure, proposed by Diewert and Wales (1987), for ensuring that a matrix of estimated parameters is negative semi-definite involves a reparameterization in which the matrix is replaced by minus the product of a *triangular* matrix and its transpose.⁸ The elements of the triangular matrix are estimated guaranteeing the desired concavity. For our purposes, the elements of B are replaced by $b_{ij} = -(DD')_{ij}$. From (8), it follows the Allen-Uzawa partials can be expressed as:

$$\Sigma = \hat{S}^{-1} \left[(-DD') + [-(\hat{S} - SS')] \right] \hat{S}^{-1} \quad (11)$$

Now provided the Hessian matrix is negative semi-definite, (11) implies Σ must also exhibit the same property. Unfortunately, imposed curvature may lead to biased elasticity estimates, *a priori* unacceptable restrictions and loss of the flexibility of the functional form (see Diewert and Wales (1987, p. 47-48, and p. 62). The upward bias induced in own- and cross- price elasticities will adversely influence the Allen-Uzawa partials in (11) as well.

B. Locally (at a reference point t^*) imposed curvature

⁸ This is the so-called Cholesky decomposition (Lau, 1978). The appropriate curvature is imposed by substituting the original parameter matrix with its Cholesky decomposition. See Featherstone and Moss (1994) for an application.

Alternatively, as discussed by Ryan and Wales (1998, 2000), curvature can be imposed locally.⁹ This “data-oriented” method selects carefully a single point to impose curvature, which then results in satisfaction at all or almost all points of the sample. This procedure has the advantage that it does not destroy the flexibility of the TL cost function.¹⁰ To impose curvature locally, set $\Gamma = -DD'$, where again D is a *triangular* matrix as above and solve for B . The elements of the parameter matrix are then replaced in estimation with the resulting matrix. This assures the concavity at the reference point. For the Allen-Uzawa partials, this implies (by (6') and (8)):

$$\Sigma = \hat{S}^{-1}[-DD']\hat{S}^{-1}. \quad (12)$$

IV. Conclusion

This note derives the following result for the TL cost function:

$$\frac{\hat{W}[\nabla_{ww}^2 C(w, y, t)]\hat{W}}{C(w, y, t)} = \hat{S}\Sigma\hat{S}. \quad \text{It proves that checking the curvature of the matrix of}$$

Allen-Uzawa partials is the same as checking the curvature of the Hessian matrix for this particular functional form. It also shows how this result extends to any functional form. In estimation, when curvature is imposed (either globally or locally), it also provides the resulting restrictions on Σ . Since the main result of this note allows an additional procedure for checking curvature, future empirical work may find it useful to study and report the matrix of Allen-Uzawa partial elasticity of substitution.

⁹ For a discussion of the differences between their method and the Cholesky decomposition, see Ryan and Wales (1998, p. 332). They also discuss several alternative methods for imposing curvature in recent literature.

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¹⁰ This follows because, after substitution for estimation purposes, the number of elements above the main diagonal in D just replaces the $(n^2-n)/2$ elements of the parameter matrix B. See also Theorem 1 in Diewert and Wales (1987, p. 46).

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Appendix:

1. Proof of the main result:

Write $\eta_{ij} = \frac{b_{ij}}{S_i} + S_j - \delta_{ij} \quad \forall i, j$, and where δ_{ij} is the Kronecker delta. Then it follows:

$$\frac{\eta_{ij}}{S_j} = \sigma_{ij} = \frac{b_{ij}}{S_i S_j} + 1 - \frac{\delta_{ij}}{S_j}, \text{ thus}$$

$$b_{ij} = S_i S_j \sigma_{ij} - S_i S_j + S_i \delta_{ij}.$$

Write the ij^{th} element of the left-hand side of (6') as:

$$\gamma_{ij} = b_{ij} - S_i \delta_{ij} + S_i S_j. \text{ Substitute for } b_{ij} \text{ to obtain:}$$

$$\gamma_{ij} = S_i S_j \sigma_{ij} \quad \forall i, j.$$

2. Negative semi-definiteness of the shares expression in (6'):

$$\begin{aligned} -[\hat{S} - SS'] &= - \left\{ \begin{bmatrix} S_1 & 0 & 0 & \dots & 0 \\ 0 & S_2 & 0 & \dots & \\ 0 & 0 & S_3 & 0 & \\ \dots & & & \dots & \\ 0 & & & & S_n \end{bmatrix} - \begin{bmatrix} S_1 S_1 & S_1 S_2 & S_1 S_3 & \dots & S_1 S_n \\ S_2 S_1 & S_2 S_2 & \dots & & \\ S_3 S_1 & \dots & S_3 S_3 & & \\ \dots & & & \dots & \dots \\ S_n S_1 & & & & S_n S_n \end{bmatrix} \right\} \\ &= - \begin{bmatrix} S_1(1 - S_1) & -S_1 S_2 & -S_1 S_3 & \dots & -S_1 S_n \\ -S_2 S_1 & S_2(1 - S_2) & & & \\ & & S_3(1 - S_3) & \dots & \dots \\ \dots & & & \dots & \dots \\ -S_n S_1 & & \dots & & S_n(1 - S_n) \end{bmatrix} \end{aligned}$$

Provided nonnegative shares ($S_i \geq 0$), this expression is negative semi-definite.