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ABSTRACT

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It is shown that any completely preordered topological real algebra admits a continuous utility representation which is an algebra-homomorphism (i.e., it is linear and multiplicative). As an application of this result, we provide an algebraic characterization of the *projective (dictatorial) preorders* defined on R^n. We then establish some welfare implications derived from our main result. In particular, the connection with the normative property called *independence of the relative utility pace* is discussed.

JEL Classification: C60, C65, D60, D63

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1 Introduction

The main purpose of this paper is to provide some new insights that explain the algebraic nature of some normative properties that naturally appear in utility theory, social choice theory and welfare economics. To that end, we present a utility theorem, in addition to some of its applications in economics, for completely preordered algebraic structures which involve more algebraic machinery than that often encountered in the (economics) literature. Namely, we pay attention to completely preordered topological real algebras, i.e., algebraic structures equipped with three binary operations and endowed with a complete and continuous preorder which is compatible with these operations. As far as we can judge, no results of representability in this context have been considered in the economics literature before. Some related material to that presented here can be found in the classical works by Birkhoff (1948) and Fuchs (1963).

We show that any completely preordered topological real algebra admits a continuous utility representation which is an algebra-homomorphism, i.e., it preserves the algebraic structure. In addition to the own algebraical interest of this result, as a consequence we obtain an algebraic characterization of the *projective preorders* defined on $\mathbb{R}^n$. If we interpret each vector $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ as the utility-vector of a society which consists of n agents, this result can be described as follows. Let $\precsim$ be a complete preorder defined on $\mathbb{R}^n$, representing the utility levels of n agents. Then $\mathbb{R}^n$, endowed with the usual operations defined *coordinatewise*, is a completely preordered topological real algebra under $\precsim$, if there is a kind of *dictator* for the society which means that the social welfare is measured just in terms of the solely welfare of that individual; that is, there is $j \in \{1, \ldots, n\}$ such that $x \precsim y \iff x_j \leq y_j$ where $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$. In other words, the existence of a dictator can be stated in terms of algebraic conditions. In the next section we also offer an interpretation of the binary operations involved in terms of welfare normative properties.

The relationship between projective preorders and what we call the *independence of the relative utility pace* (I.R.U.P.) property is also discussed. This condition can be described as follows. Let $x, y \in \mathbb{R}^n$ be two utility-vectors of an n-agents society. Then, $x = (x_1, \ldots, x_j, \ldots, x_n) \precsim y = (y_1, \ldots, y_j, \ldots, y_n)$ iff $(x_1, \ldots, f(x_j), \ldots, x_n) \precsim (y_1, \ldots, f(y_j), \ldots, y_n)$ for every $j \in \{1, \ldots, n\}$, and every increasing map $f : \mathbb{R} \longrightarrow \mathbb{R}$. In words, I.R.U.P. means that individual increase of utility levels does not necessarily increase the welfare of society. Intuitively, this loss of welfare can be imputed to the existence of dominating coalitions. In fact, we prove that if $\precsim$ is non-trivial and increasing then the independence of the relative utility pace assumption implies the existence of a dictator. We conclude with a remark explaining the role that algebraic issues of the kind considered in this note play in *social choice theory.*

2 Previous definitions and notations

Throughout the paper, $\precsim$ will denote a complete preorder defined on a (non-empty) set X, i.e., a binary relation on X which is *reflexive*, ($x \precsim x$ for all $x \in X$), *transitive* ($x \precsim y$, $y \precsim z$ implies $x \precsim z$); and *decisive* or *complete* (either $x \precsim y$ or $y \precsim x$ for every $x, y \in X$). We also say that $(X, \precsim)$ is a *completely preordered set*. Two elements $x, y \in X$ are indifferent if $x \precsim y$ and $y \precsim x$ (briefly, $x \sim y$). If, in addition, $\precsim$ is anti-symmetric ($x \precsim y$ and $y \precsim x$ implies $x = y$) then $\precsim$ is said to be a *total order*.

Given $\precsim$, we consider as usual the *strict binary relation* $\prec$ on X defined as: $x \prec y$ iff no ($y \precsim x$). The preorder $\precsim$ is said to be *non-trivial* if there are $x, y \in X$ such that $x \prec y$.

Let $(X, \precsim)$ be a completely preordered set, a function $u : X \longrightarrow \mathbb{R}$ is said to be a *utility function* for $\precsim$ if, for every $x, y \in X$, it holds that $x \precsim y \Longleftrightarrow f(x) \leq f(y)$.

We will focus our attention on spaces X having a rich algebraic structure, namely, rings and real algebras. For the sake of completeness, we only recall the definition of a real algebra. [1]

Definition 2.1. A *real algebra* $(X, +, \cdot_{\mathbb{R}}, *)$ is a space X endowed with three binary operations such that:

(i) $(X, +, \cdot_{\mathbb{R}})$ is a real vector space.

(ii) $(X, +, *)$ is a ring.

(iii) $\lambda \cdot (x * y) = (\lambda \cdot x) * y = x * (\lambda \cdot y)$, $\forall x, y \in X$, $\forall \lambda \in \mathbb{R}$.

Let $(X, +, \cdot_{\mathbb{R}}, *)$ be a real algebra. A function $v : X \longrightarrow \mathbb{R}$ is said to be an *algebra-homomorphism* if it is linear and multiplicative, i.e.:

(a) $v(\lambda \cdot x + \beta \cdot y) = \lambda v(x) + \beta v(y)$, $\forall x, y \in X$, $\forall \lambda, \beta \in \mathbb{R}$.

(b) $v(x * y) = v(x)v(y)$, $\forall x, y \in X$.

Definition 2.2. A *completely preordered real algebra* $(X, \precsim, +, \cdot_{\mathbb{R}}, *)$ is a real algebra equipped with a complete preorder $\precsim$, which is compatible with the operations $+, \cdot_{\mathbb{R}}$ and $*$, i.e.:

(1) $x \precsim y$ implies $x + z \precsim y + z$, $\forall z \in X$.

(2) $x \precsim y$, $0 \leq \lambda$ implies $\lambda \cdot x \precsim \lambda \cdot y$.

(3) $x \precsim y$, $\mathbf{0} \precsim z$ imply $z * x \precsim z * y$ and $x * z \precsim y * z$ ($\mathbf{0}$ denotes the null element with respect to $+$).

[1] The reader will find the textbook of Hungerford (2003) a useful reference for algebraic issues.

Following Moulin(1988) let us provide some welfare interpretations of axioms (1), (2) and (3) above. Condition (1) says that $\precsim$ is *zero independent* which means that comparisons between utility streams depend upon absolute variations of utilities or, equivalently, $\precsim$ is invariant to independent changes of origins. Condition (2) states that $\precsim$ is *homothetic* (see, e.g., Candeal and Induráin (1995 a)) which reflects the independence of a common utility scale (or common relative variations of utility). Finally, condition (3) means that $\precsim$ is *independent of arbitrarily relative variations of utilities* or, equivalently, it is invariant to independent changes of units (in which, say, utility is measured). Of course, all of these welfare interpretations for $\precsim$ are easily established provided that $X = \mathbb{R}^n$ and the operations $+, \cdot_{\mathbb{R}}$ and $*$ are the usual ones defined coordinatewise. However, they are still meaningful in broader contexts like l_∞, the space of bounded real sequences, which naturally arises in intertemporal decision problems.

As we are interested in obtaining strong properties of representability for $\precsim$, we allow X to be a topological space endowed with a topology τ. In order for this to be made in a natural way, we require $(X, \tau, +, \cdot_{\mathbb{R}})$ to be a *topological vector space* (note that no assumptions involving topological considerations are made on $*$). We are now ready to introduce a key definition.

Definition 2.3. We say that $(X, \precsim, \tau, +, \cdot_{\mathbb{R}}, *)$ is a *completely preordered topological real algebra* if

(i) $(X, \precsim, +, \cdot_{\mathbb{R}}, *)$ is a completely preordered real algebra.

(ii) $(X, \tau, +, \cdot_{\mathbb{R}})$ is a topological real vector space.

(iii) The preorder $\precsim$ is *continuous*, i.e., for every $x \in X$, the lower and upper contour sets $L(x) = \{y \in X; y \precsim x\}$ and $G(x) = \{y \in X; x \precsim y\}$, respectively, are closed in X (i.e., the complements of $L(x)$ and $G(x)$ belong to τ for every $x \in X$).

A complete preorder $\precsim$ defined on a subset $X \subseteq \mathbb{R}^n$ is said to be *increasing* if for every $x, y \in X$, such that that $x \leq y$ it holds that $x \precsim y$, where by $x \leq y$ we mean $x_i \leq y_i$, $\forall i \in \{1, \ldots, n\}$, $x = (x_1, \ldots, x_i, \ldots, x_n)$, $y = (y_1, \ldots, y_i, \ldots, y_n)$.

Let us state now the so-called in the mathematical utility literature Eilenberg-Debreu theorem which will be used later on.

Theorem 2.4. *(Eilenberg(1941), Debreu(1964)) Let (X, τ) be a second countable topological space. Then every τ-continuous complete preorder $\precsim$ defined on X can be represented by a continuous utility function.*

3 The main result

In this section we obtain a theorem of representability of a completely preordered topological real algebra by means of a continuous utility function which is an algebra-homomorphism, i.e., it is linear and multiplicative. It is important to say that the result is also valid in an infinite-dimensional context and nothing about the commutativity of the binary operation $*$ is assumed.

Before stating the theorem, we need a preliminary lemma.

Lemma 3.1. *Suppose that* $\mathbb{R}$ *is endowed with both a binary operation* $*$ *and a non-trivial continuous total order* $\precsim$*, in such a way that* $(\mathbb{R}, \precsim, \tau_{Euclidean}, +, *)$ *becomes a completely preordered topological ring* [2]. *Then* $\precsim$ *is representable by a continuous utility function which is a ring-homomorphism. Moreover, there exists* $a \in \mathbb{R} \setminus 0$*;* $x * y = axy$*,* $\forall x, y \in \mathbb{R}$*.*

Proof. Since $\precsim$ is continuous on $\mathbb{R}$, it follows from Theorem 2.4 that there exists an (injective) continuous utility function $u : \mathbb{R} \longrightarrow \mathbb{R}$ for $\precsim$. Actually, u is a homeomorphism. Thus, without loss of generality, we can assume that either $u(x) = x$ or $u(x) = -x$ $(x \in \mathbb{R})$. Suppose that the first holds, the other case being analyzed in a similar way.

Now by a result by Pickert and Hion (see Birkhoff (1948), p. 398-399), $\precsim$ is representable by a utility function $v : \mathbb{R} \longrightarrow \mathbb{R}$ which is a ring-homomorphism. So, because u and v are utility functions for $\precsim$, there exists an increasing function $g : v(\mathbb{R}) \longrightarrow \mathbb{R}$, such that $gv(x) = x$, $\forall x \in \mathbb{R}$. Let us see that v is also increasing. For that, let $x, y \in \mathbb{R}$; $x < y$. Then, because v is injective, either $v(x) < v(y)$ or $v(y) < v(x)$. If $v(y) < v(x)$ then, since g is increasing, $gv(y) < gv(x)$, i.e., $y < x$, which is a contradiction. So, $v(x) < v(y)$, and v is an increasing function. Remember that v is an additive function and it is well known that if v is discontinuous at some point then it is discontinuous at every point of $\mathbb{R}$. This last fact, being v increasing, leads to $v(x) = ax$ (with $a > 0$). Because $a(x*y) = v(x*y) = v(x)v(y) = a^2xy$, this type of representation gives $x*y = axy$ $\forall x, y \in \mathbb{R}$. □

Remark 3.2. As the lemma shows, the operation $*$ is essentially the usual multiplicative one in $\mathbb{R}$. Note that $(\mathbb{R}, +, *)$ is a *field* and the map $(x, y) \in \mathbb{R} \times \mathbb{R} \longrightarrow x * y \in \mathbb{R}$ is continuous.

Let us now prove the main result of the paper, which involves a very particular kind of utility representation for completely preordered topological real algebras.

Theorem 3.3. *Let* $(X, \precsim, \tau, +, \cdot_{\mathbb{R}}, *)$ *be a completely preordered topological real algebra. Then* $\precsim$ *admits a continuous utility function* $\Gamma : X \longrightarrow \mathbb{R}$ *which is, in addition, an algebra-homomorphism.*

Proof. If $\precsim$ is trivial, then $\Gamma \equiv 0$ works. Otherwise, consider the set $I(\mathbf{0}) = \{x \in X;\ x \sim \mathbf{0}\}$. First, let us see that $I(\mathbf{0})$ is an ideal of X, for which we need to prove the following two properties:

(i) $I(\mathbf{0})$ is a real vector subspace of X.

[2] $(X, \precsim, +, *)$ is a completely preordered ring if $(X, +, *)$ is a ring and $\precsim$ is a complete preorder on X which satisfies condition (1) and (3) of Definition 2.2

(ii) For every $x \in I(\mathbf{0})$, $y \in X$, it holds that $y * x \in I(\mathbf{0})$ and $x * y \in I(\mathbf{0})$.

Let $x, y \in I(\mathbf{0})$. Because $\precsim$ is compatible with $+$, it follows that $x+y \sim x+\mathbf{0} = x \sim \mathbf{0}$. So, in order to prove (i), it is enough to see that, given $x \in I(\mathbf{0})$ and $\lambda \in \mathbb{R}$, then $\lambda \cdot x \sim \mathbf{0}$. To that end, denote by $S(x)$ the linear subspace generated by x, i.e, $S(x) = \{\lambda \cdot x;\ \lambda \in \mathbb{R}\}$, and consider the preorder $\precsim$ restricted to $S(x)$. Since $(X, \tau, +, \cdot_{\mathbb{R}})$ is a topological vector space, endowed with the relative topology of X, $S(x)$ is homomorphic to $\mathbb{R}$ and therefore it is a connected and separable topological space. Because $\precsim$ is continuous it follows, by Theorem 2.4, that there exists a continuous utility function $u : S(x) \longrightarrow \mathbb{R}$ which represents $\precsim$. By identifying $S(x)$ with $\mathbb{R}$, the function u clearly satisfies the following functional equation $u(\lambda_1) = u(\lambda_2)$ iff $u(\lambda_1 + \alpha) = u(\lambda_2 + \alpha)$, $\forall \alpha \in \mathbb{R}$ or, equivalently, the function $s \in \mathbb{R}_{++} \longrightarrow v(s) = u(logs)$ satisfies the functional equation: $v(s_1) = v(s_2)$ iff $v(\lambda s_1) = v(\lambda s_2)$, $\forall \lambda \in \mathbb{R}_{++}$. It was shown in Candeal and Induráin (1993) that the only continuous solutions of the latter functional equation are the constant functions. Thus, every element in $S(x)$ is indifferent to $\mathbf{0}$.

To prove (ii), let $y \in X$ and $x \in I(\mathbf{0})$. If $\mathbf{0} \precsim y$ then, since $(X, +, \cdot_{\mathbb{R}}, *)$ is a completely preordered real algebra, $y * x \precsim y*\mathbf{0} = \mathbf{0}$ and $\mathbf{0} = y*\mathbf{0} \precsim y * x$. Therefore, $y * x \sim \mathbf{0}$. If $y \precsim \mathbf{0}$, then $\mathbf{0} \precsim -y$ and so $-y * x \sim \mathbf{0}$. Because $y * x = -(-y) * x$, and $I(\mathbf{0})$ is a vector subspace of X, it holds that $y * x \sim \mathbf{0}$. Similarly, we can prove that $x * y \sim \mathbf{0}$. Thus (ii) holds and therefore $I(\mathbf{0})$ is an ideal of X.

Now consider the quotient space $X/I(\mathbf{0})$. Because $X/I(\mathbf{0})$ coincides with the quotient space $X/\sim$, $X/I(\mathbf{0})$ is a totally ordered set. We denote the natural order on $X/I(\mathbf{0})$ by $\precsim'$. Since $I(\mathbf{0})$ is an indeal of X, the operations $+, \cdot_{\mathbb{R}}$ and $*$ pass to the quotient in such a way that $(X/I(\mathbf{0}), \precsim', +, \cdot_{\mathbb{R}}, *)$ becomes a totally ordered real algebra. Moreover, endowed with the quotient topology ς, $X/I(\mathbf{0})$ is a topological vector space because $I(\mathbf{0})$ is a closed subspace of X which follows from the fact that $\precsim$ is continuous. Therefore, $(X/I(\mathbf{0}), \precsim', \varsigma, +, \cdot_{\mathbb{R}}, *)$ is a totally ordered topological real algebra. By Theorem 2.4 again, a standard connectedness argument shows that the co-dimension of $I(\mathbf{0})$ is one or, equivalently, $X/I(\mathbf{0})$ is homeomorphic and isomorphic as a vector space to $\mathbb{R}$. Now, by the previous lemma, $\precsim'$ is representable by a continuous utility function $u : X/I(\mathbf{0}) \longrightarrow \mathbb{R}$, which is linear and multiplicative. Let us denote by $p : X \longrightarrow X/I(\mathbf{0})$ the projection map. It is easy to see that p is linear, multiplicative and continuous. Then it is enough to consider the composition $u \circ p : X \longrightarrow \mathbb{R}$ to obtain the desired result. □

4 Applications

In this section we provide some applications of our theorem. The first one is concerned with the characterization of all (continuous) complete preorders $\precsim$ defined on $\mathbb{R}^n$ for which $(\mathbb{R}^n, \precsim, \tau_{Euclidean})$, under the usual operations defined coordinatewise, is a completely preordered topological real algebra. This will lead us to the notion of *projective* preorder. The second one links the algebraic condition of being a completely preordered topological real algebra with a normative property; namely, the *independence of the relative utility pace*, which turns out to be a generalization of the property known as the *independence of the common utility pace* introduced by D'Aspremont and Gevers (1977). For a survey of this, and other normative properties and their welfare significance see Moulin (1988) and, more recently, Bossert and Weymark (2004) and Fleurbaey and Hammond (2004).

Definition 4.1. Let $\precsim$ be a complete preorder defined on $\mathbb{R}^n$ and let $j \in \{1, \ldots, n\}$. We say that $\precsim$ is *j-projective* if for every $x = (x_1, \ldots, x_j, \ldots, x_n)$, $y = (y_1, \ldots, y_j, \ldots, y_n)$, it holds that $x \precsim y$ if and only if $x_j \leq y_j$. $\precsim$ is said to be *projective* if it is j-projective for some $j \in \{1, \ldots, n\}$.

Let $\mathbb{R}^n$ be endowed with the usual operations $+, \cdot_{\mathbb{R}}$ and $*$ defined coordinatewise. It is then clear that $(\mathbb{R}^n, +, \cdot_{\mathbb{R}}, *)$ is a real algebra. Now, we are interested in finding all complete preorders $\precsim$ defined on $\mathbb{R}^n$ for which $(\mathbb{R}^n, \precsim, \tau_{Euclidean}, +, \cdot_{\mathbb{R}}, *)$ becomes a completely preordered topological real algebra.

Proposition 4.2. *The only complete preorders $\precsim$ defined on $\mathbb{R}^n$ for which $(\mathbb{R}^n, \precsim, \tau_{Euclidean}, +, \cdot_{\mathbb{R}}, *)$ is a completely preordered topological real algebra are the projective ones.*

Proof. Let $\precsim$ be a preorder defined on $\mathbb{R}^n$ such that $(\mathbb{R}^n, \precsim, \tau_{Euclidean}, +, \cdot_{\mathbb{R}}, *)$ is a completely preordered topological real algebra. By using the representation result (Theorem 3.3), it follows that there is a continuous, linear and multiplicative function $\psi : \mathbb{R}^n \longrightarrow \mathbb{R}$ which represents $\precsim$. Because ψ is linear and continuous, ψ is of the form $\psi(x_1, \ldots, x_n) = \sum_{i=1}^{n} a_i x_i$, for some $a_i \in \mathbb{R}$, $i = 1, \ldots, n$. Let us prove that all, but at most one, coefficients a_i are zero. Suppose by way of contradiction that there exist $i \neq j$ such that $a_i \cdot a_j \neq 0$. Then, because ψ is also multiplicative, $\psi(0, \ldots, 0, \ldots, 0) = \psi((0, \ldots, 1_i, \ldots, 0) * (0, \ldots, 1_j, \ldots, 0))= \psi(0, \ldots, 1_i, \ldots, 0)\psi(0, \ldots, 1_j, \ldots, 0)$. That is, $0 = a_i \cdot a_j$. This contradiction proves that there is $j \in \{1, \ldots, n\}$ such that $\psi(x_1, \ldots, x_n) = a_j x_j$. It should be noted in addition that $\psi((1, \ldots, 1) * (1, \ldots, 1)) = \psi(1, \ldots, 1) = \psi(1, \ldots, 1)\psi(1, \ldots, 1)$. In other words, $a_j = a_j^2$, hence $a_j = 0$ or 1. Thus, if $a_j = 0$, then $\precsim$ is trivial (all the utility vector are indifferent) and $\precsim$ is j-projective, for all $j \in \{1, \ldots, n\}$. If $a_j = 1$, then $\psi(x_1, \ldots, x_n) = x_j$ and $\precsim$ is a j-projective complete preorder defined on $\mathbb{R}^n$. □

Remarks 4.3. (i) Proposition 4.2 can be re-stated in the following equivalent terms: *The only complete and continuous preorders defined on $\mathbb{R}^n$ which satisfy zero-independence, homotheticity and independence of arbitrarily relative variations of utilities are the projective ones.*

(ii) It should be noted that the conclusion of the previous proposition need not be true in the infinite-dimensional context. For example, if the "utility vectors" represent streams of utility with (a countable number of) infinite coordinates like in the intertemporal decision problems, there could exist non-projective continuous complete preorders defined on the space of utility streams in such a way that this space can be given a structure of a completely preordered topological real algebra. This is the case for the space l_∞, which consists of all real bounded sequences, and that is often encountered in the economics literature related to decision problems over time involving an infinite horizon. It is easy to check that with the usual operations defined coordinatewise, $(l_\infty, \|\cdot\|_\infty, +, \cdot_{\mathbb{R}}, *)$ is a topological real algebra (in fact, it is a Banach real algebra); the topology given by the supremum norm $\|x\|_\infty = \|(x_1, \ldots, x_n, \ldots)\|_\infty = sup\{|x_n|; n \in \mathbb{N}\}$. Let us denote by $\beta(\mathbb{N})$ the Stone-Čech compactification of the set of the natural numbers $\mathbb{N}$ (see Dugundji (1966)). Let $p \in \beta(\mathbb{N}) \setminus \mathbb{N}$. Then the evaluation map at p defines a continuous, linear and multiplicative map $x \in l_\infty \rightsquigarrow e_p(x) \in \mathbb{R}$. In other words, e_p belongs to the spectrum of the Banach algebra $(l_\infty, \|\cdot\|_\infty, +, \cdot_{\mathbb{R}}, *)$ (see, Rickart, (1960)). Thus, the complete preorder $\precsim_p$ defined on l_∞ as $x \precsim_p y \Leftrightarrow e_p(x) \leq e_p(y)$ is continuous and allows the structure $(l_\infty, \precsim_p, \|\cdot\|_\infty, +, \cdot_{\mathbb{R}}, *)$ to become a completely preordered topological real algebra. It should be noted that, from the point of view of an economic interpretation, $\precsim_p$ can be considered as the existence of an *invisible dictator or social planner.*

(iii) Although the conclusion of Proposition 4.2 does not hold, in general, in the infinite-dimensional setting (see Remarks (ii) above), it can still be generalized. Indeed, the statement remains true for the space of real sequences which vanish at infinity, usually denoted by c_0. Equipped with the usual operations defined coordinatewise and the topology given by the supremum norm, c_0 is a topological real algebra (in fact, a Banach subalgebra of l_∞). The reason for the statement of Proposition 4.2 to be true in this case is that the spectrum of c_0, the set of all linear and multiplicative functionals defined on it, can be easily seen to coincide with the set of natural numbers $\mathbb{N}$.

Definition 4.4. Let $\precsim$ be a complete preorder defined on $\mathbb{R}^n$. We say that $\precsim$ is *independent of the relative utility pace* (briefly, I.R.U.P.) if $(x_1, \ldots, x_j, \ldots, x_n) \precsim (y_1, \ldots, y_j, \ldots, y_n)$ implies $(x_1, \ldots, f(x_j), \ldots, x_n) \precsim (y_1, \ldots, f(y_j), \ldots, y_n)$ for every $j \in \{1, \ldots, n\}$ and every increasing map $f : \mathbb{R} \longrightarrow \mathbb{R}$.

On the basis of this definition we can state the following result.

Proposition 4.5. *Let $\precsim$ be a non-trivial increasing ($x \leq y$ implies $x \precsim y$) and continuous complete preorder defined on $\mathbb{R}^n$. Then the following statements are equivalent:*

(i) $\precsim$ is projective

(ii) $\precsim$ is I.R.U.P.

Proof. (i)⇒(ii) is obvious. (ii)⇒(i). By induction on n. For $n = 1$ it is obvious since $\precsim$ is the usual ordering on $\mathbb{R}$. Let us prove it for $n = 2$. Let $z = (z_1, z_2)$, $x = (x_1, x_2)$ and $y = (y_1, y_2) \in \mathbb{R}^2$. Because $\precsim$ is I.R.U.P., we have that $(x_1, x_2) \precsim (y_1, y_2)$ if and only if $(x_1 + z_1, x_2) \precsim (y_1 + z_1, y_2)$ by taking the increasing function $f(r) = r + z_1$ $(r \in \mathbb{R})$. By repeating this process over the second coordinate, we obtain $x \precsim y \Leftrightarrow x + z \precsim y + z$, i.e., $\precsim$ is compatible with respect to + (property (1) of Definition 2.2). This fact, together with $\precsim$ being continuous, implies that $\precsim$ can be represented by a linear utility function (see Trockel, (1992) or Candeal and Induráin (1995 b)). Since $\precsim$ is non-trivial and increasing, there is a non-null $\mathbf{0} \leq p = (p_1, p_2) \in \mathbb{R}^2$ such that $x \precsim y \Leftrightarrow (p, x) \leq (p, y)$, where $(\cdot)$ denotes the usual inner product in $\mathbb{R}^n$. Let $\mathbf{0} \neq z = (z_1, z_2) \sim (0, 0)$. Because the set $I(\mathbf{0})$ is a linear subspace and $\precsim$ is increasing, we can have neither $z_1, z_2 > 0$ nor $z_1, z_2 < 0$. Suppose now, without loss of generality, that $z_1 \geq 0$, $z_2 \leq 0$. If $z_1 > 0$, then by taking an increasing function $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that $f(z_1) = k$ $(k > 0)$ and $f(0) = 0$, by I.R.U.P., we have that $(f(z_1), z_2) \sim (f(0), 0)$; i.e., $(k, 0) \sim (0, 0)$. So $p_1 = 0$ (and therefore $z_2 = 0$), hence $\precsim$ is 2-projective. A similar argument holds for $z_2 < 0$. In any case, $\precsim$ is projective.

To show the general case, note first that, by the same arguments as for $n = 2$, there exists a non-null $\mathbf{0} \leq p = (p_1, \ldots, p_n) \in \mathbb{R}^n$ such that $x \precsim y \Leftrightarrow (p, x) \leq (p, y)$ $(x, y \in \mathbb{R}^n)$. Consider $\mathbb{R}^{n-1} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n; x_n = 0\}$. If $\precsim$ is trivial on $\mathbb{R}^{n-1}$, then $p_1 = \ldots = p_{n-1} = 0$ and $p_n \neq 0$. Then, $\precsim$ is n-projective. Suppose, otherwise, that $\precsim$ is non-trivial on $\mathbb{R}^{n-1}$. Then, by induction hypothesis (note that the continuity, increasingness and I.R.U.P. are induced over linear subspaces), there is $j \in \{1, \ldots, n-1\}$ such that $p_j \neq 0$ and $p_i = 0$ $\forall i \in \{1, \ldots, n-1\} \setminus \{j\}$. Now, it is enough to consider $\mathbb{R}^{j,n} = \{(0, \ldots, x_j, 0, \ldots, x_n); x_j, x_n \in \mathbb{R}\}$ and apply again the induction hypothesis for $n = 2$ to obtain that, in this case, necessarily $p_n = 0$. Thus, $\precsim$ is j-projective. $\square$

The next corollary follows immediately and shows an algebraic characterization of those complete preorders defined on $\mathbb{R}^n$ which are independent of the relative utility pace. Assume that $\mathbb{R}^n$ is endowed with the usual operations defined coordinatewise.

Corollary 4.6. *Let $\precsim$ be a non-trivial increasing preorder defined on $\mathbb{R}^n$. Then $(\mathbb{R}^n, \precsim, \tau_{Euclidean}, +, \cdot_{\mathbb{R}}, *)$ is a completely preordered topological real algebra if and only if $\precsim$ is I.R.U.P.*

Remark 4.7. In this paper we have presented a mathematical result concerning the existence of continuous linear and multiplicative utility for the class of ordered structures called *completely preordered topological real algebras*. In particular, we have offered an algebraic characterization of the class of social welfare orderings called *projective preorders* defined on $\mathbb{R}^n$. Since this latter class plays an important role in social choice theory, it seems natural to ask whether or not the approach followed can result into a different methodology to obtain, say, Arrow-like theorems. Indeed, this is not new in the literature. It is well known (see, e.g., D'Aspremont and Gevers (1977)) that a social welfare functional, defined on profiles of individual preference orderings, induces a social welfare functional defined on profiles of individual utilities. Moreover, independence properties of the former rules pass to invariance properties of the latter. This process yields social welfare functionals that can be represented by social welfare orderings. Now, it would be expected that our algebraic characterization of the dictatorial preorders (defined even in more general settings than $\mathbb{R}^n$, see Remarks 4.3, (iii)) can result into a characterization of dictatorial social welfare functionals, say, à la Arrow. The nub point here is the continuity of the social welfare ordering induced by the social welfare functional.

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