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Empirical Challenges in the Capability Approach: Measuring Capability Sets and Unfreedom through Counterfactual Comparisons

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Empirical Challenges in the Capability Approach: Measuring Capability Sets and Unfreedom through Counterfactual Comparisons^{*}

Abstract

This study addresses a fundamental challenge in the empirical application of the Capability Approach: the measurement of the “capability set” as an opportunity set. Unlike standard utility-based measures that focus solely on achieved outcomes, measuring capability requires assessing the welfare of potential activities—including those not chosen (counterfactuals). We propose a novel methodology that bridges normative social choice theory and econometric causal inference. Specifically, we interpret the Average Treatment Effect (ATE) derived from panel data fixed-effects models as capturing marginal counterfactual welfare differences between alternative actions, rather than level comparisons of achieved outcomes. Using a unique panel dataset of elderly individuals in Japan, focusing on “going-out” versus “staying-home” behavior, we evaluate the size of capability sets and the degree of “unfreedom” (the welfare gap between options). Furthermore, we propose and apply several aggregation rules—ranging from Utilitarian to Rawlsian—to construct group-level capability measures. Our empirical results demonstrate that the ranking of social groups varies significantly depending on the normative aggregation rule employed, highlighting the importance of explicitly defining the informational basis of social evaluation.

JEL classification

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Keywords

capability approach, causal inference, average treatment effect, social choice theory, unfreedom

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1 Introduction

1.1 The purpose of this paper: Applying capability approach in empirical economics

The Capability Approach (CA), pioneered by Amartya Sen, has profoundly influenced welfare economics by shifting the focus from traditional measures of utility (satisfaction) or resources (income) to the actual freedoms individuals have to achieve different ways of life (i.e. bundles of functionings) they value. Despite its theoretical prominence, the empirical application of the CA remains quite limited.

The central methodological challenge — and the focus of this article — lies in the core requirement of the CA: measuring the capability set as an opportunity set. A fundamental tenet of the CA is that welfare evaluation should consider not only the achieved functionings (the state of being or doing) but also their freedom to choose among feasible alternatives. Empirically, this implies that assessing an individual’s welfare requires quantifying the potential value or welfare associated with activities that were not actually chosen. In econometric terms, this corresponds to the problem of evaluating counterfactual outcomes¹.

This article proposes a unified methodology to address this core measurement problem of the CA by synthesizing causal inference from empirical economics with axiomatic social choice theory. In other words, we draw an explicit parallel between the requirement to measure the full capability set and the “fundamental problem of causal inference” — the impossibility of observing both outcomes (treated and untreated) for the same individual at the same time. Specifically, we argue that the Average Treatment Effect (ATE), estimated via individual fixed effects in panel data, provides a coherent normative basis for constructing the opportunity set. By exploiting within-individual variation over time, the fixed-effects estimator allows us to axiomatically reconstruct the welfare associated with unchosen options by aggregating experiences across time.

We apply this framework to a unique panel dataset of elderly individuals in Japan, focusing on a binary choice of “going out” versus “staying home.” Under the assumption that the individual fixed effects estimator identifies the ATE in terms of functionings, we develop a theoretical framework for aggregating individual capability measures into a social evaluation, considering alternative normative aggregation rules, including, among others, Utilitarian and Rawlsian criteria.

Our analysis illustrates that the choice of normative criteria has substantive implications for measured capability sets related to going-out behavior across groups. For instance, when greater weight is placed on decomposability criterion at the group level, individuals with disabilities are assessed as having substantially larger capability sets than public care recipients. Under this normative criterion, the levels of unfreedom — defined as the shortfall between potential and realized functionings — are evaluated as being broadly comparable between two groups. In contrast, when impartiality across types is emphasized, the measured

¹ Existing empirical indices inspired by the Capability Approach, such as the Human Development Index (HDI), have played an important role in moving welfare analysis beyond uni-dimensional measures. Nevertheless, by construction, these indices necessarily focus on realized achievements rather than the full set of feasible alternatives available to individuals. <https://hdr.undp.org/data-center/human-development-index#/indicies/HDI>

sizes of capability sets for the two groups are nearly identical, while the level of unfreedom is assessed to be larger for individuals with disabilities than for public care recipients. These differences in normative aggregation can, in turn, influence the prioritization of policies aimed at facilitating going-out behavior, such as subsidies for public transportation, across population groups.

Several limitations of our approach should be noted. First, in the CA, an individual’s well-being is typically captured as a multidimensional vector of functionings, but there remains arguments how to summarize such multidimensionality. In this article, we use an ad hoc set of multidimensional functionings relevant to our subject and context, and then we propose mapping them into two functioning spaces corresponding to mutually exclusive actions: “going out” versus “staying home.” Second, our approach assumes that the individual fixed-effects estimator consistently identifies the relevant ATEs. Potential sources of bias arising from violations of this assumption are not addressed in the present analysis and remain an important topic for future research.

Despite these limitations, our approach allows us to quantify both the size of individual capability sets and the degree of unfreedom, defined as the welfare disparity among the options available to an individual.

1.2 Relationship to Prior Research

1.2.1 Two Main Streams in Measuring “Opportunity”

The previous research on measuring “opportunity” can be broadly divided into two main streams. One stream focuses on “equality of opportunity” as its key concept, while the other stream centers on “opportunity as freedom.” Both streams share a common understanding: “opportunity” refers to the set of choices available to an individual prior to making a specific choice, meaning they could have chosen any option within that set. They also share the common social goal of respecting individual freedom while striving for equality among individuals. However, they differ in several important respects.

The first strand of research that centered on “equality of opportunity” includes three groups. The first includes social choice studies that characterize different allocation rules through binary axioms (e.g. circumstance vs. responsibility, compensation vs. reward) and expose contradictions among axioms in each of allocation rule (Fleurbaey, 1994, 1995; Fleurbaey and Maniquet, 2012). The second group includes analytical Marxist studies that, starting from inequality of outcomes (e.g. income), discount the contribution attributable to individual responsibility (freedom or preference/choice) and retain only the contribution attributable to environment (Roemer, 1998). The third group consists of philosophical theories, which argue that equalizing outcomes without excluding the contribution of individual responsibility is both unfair and potentially freedom-reducing (Dworkin, 1981b,a; Arneson, 1989). As discussed later, this is the field where empirical research has advanced the most (Fleurbaey et al., 2017; Van de gaer and Ramos, 2020).

On the other hand, the second strand of research that centered on the keyword “freedom as opportunity” is grounded in Sen’s CA, which holds that capturing individual freedom (and its constraints) requires assessing the goodness of the opportunity set beyond the goodness of outcomes. While methods for comparatively evaluating freedom were explored by combining

foundational axioms as in the above first strand of literature, the literature on freedom-based opportunity evaluation can be broadly categorized into two types: volume and quality (Pattanaik and Xu, 2024) ².

Volume assumes cardinal measurability based on element count or distance (for example, Pattanaik and Xu (1990); Kolm (2010); Miyagishima (2010); Gaertner and Xu (2008, 2011); Dhongde et al. (2019)). Quality assumes ordinal or cardinal comparability based on individual or societal preferences such as the intersection of indirect utility functions, social welfare functions, or plural indirect utility functions (for example, Sen (2009); Pattanaik and Xu (1998)) ³.

It is worth noting that both strands of the literature stem from a common methodological interest in extending the informational foundations of traditional economics. Specifically, Gary Becker and Kelvin Lancaster expanded the domain of the utility function from the “goods and resources” space to the space of “characteristics” that includes economic and non-economic factors (Lancaster, 1966) or to the space of “commodities” that are abstractly obtained by transforming goods and resources (Becker, 1965) (reprinted in 1967). However, their focus remained confined to vector points, rather than to the opportunity set. Sen’s proposed CA not only extended the domain of the utility function (which he calls the evaluation function) to the space of “functionings,” but also posited the “opportunity set” itself as the informational foundation (Sen, 1980, 1985, 1999a, 2016). Building on Sen’s approach, (Herrero, 1996) redefines distributive justice rules in terms of capabilities, and Gotoh and Yoshihara (2003) constructed a distributive rule using the common subset of individuals’ opportunity sets as the least advantage within society. Our definition of capability set is based on this literature.

1.2.2 The methodological challenge of measuring capability

Following Sen’s CA, freedom is understood as the quality of the opportunity set itself — what Sen refers to as the opportunity of freedom. In contrast, Ferreira and Peragine (2016), working within the equality of opportunity tradition, refer to this perspective as “opportunity as liberty” and emphasize the empirical difficulty arising from the unobservability of opportunity sets. The set of opportunities is inherently unobservable; by definition, it is a set of counterfactual choices ⁴.

They recommend a method for indirectly inferring the set of opportunities available to an individual based on observable elements, or elements inferable from observable characteristics (such as the distribution of environment, effort, and outcomes that can be statistically

² The evaluating rule for the goodness of the opportunity set is also known as “opportunity ranking” in the field.

³ The attempt to focus on quality is formally related to the “preference for flexibility” under uncertainty presented by Kreps (1979). Amartya Sen, however, interprets Kreps’s preference for flexibility as reducing the value of opportunity or freedom to the single value of the best choice, and he questions that “His problem is not that of measuring opportunity-freedom where multiple preferences relate to a person’s autonomy and are ultimately open to that person’s own choice.” For example, “(t)he addition of an inferior alternative does not worsen the persons’ freedom, though it would tend to worsen the situation in the case of rational choice under uncertainty.” (Sen, 2002, p. 619.)

⁴ This observation, like that of the budget set, echoes Basu (1987), who pointed out that depending on the choice of the outcome, the opportunity set itself may change. With Basu’s critique, Bossert et al. (1994) and Pattanaik and Xu (2024) proceeded to a game-theoretic analysis.

inferred from age, gender, and other observable characteristics of data). As an example, Ferreira and Peragine (2016) introduce the fixed effects estimation method of Niehues and Peichl (2014), aiming to decompose the actual income inequality into the contribution of circumstances (fixed effects) and the contribution of individual’s efforts (observable explanatory variables).

1.3 Conceptual Framework: aggregation based on group identity

Our argument also attempts to connect with statistics in pursuit of empirical research on measuring “opportunities as freedom.” Rather than interpreting the unobservable elements as proxied by observable elements, we refer the statistical concept of the fundamental problem of causal inference, and explore methods to capture the entire set of opportunities, including unobservable choices.

First, the fundamental problem of causal inference refers to the impossibility of comparing the observed outcome of an individual who took a certain action with the counterfactual outcome that would have occurred had the individual taken a different action at the same point in time. As well known in economics, for this problem, Rubin (1974), for example, proposed a method of capturing it using the difference between the mean values of the treatment group and the control group.

Second, we interpreted such a statistical method to estimate treatment effects as a method of inferring the points an individual has not actually achieved, based on the points realistically realized by group members who are assumed to share common conditions. Therefore, in this paper, an individual’s capability is interpreted as the accumulation of points achieved by group members. This involves a shift in perspective from an “individual identity” to a “group identity” based on a certain “group consciousness”⁵.

Third, more specifically, this paper employs two key concepts: *group* and *type*. A group refers to a collection of individuals who share a common set of available resources and utilization abilities. The term “group consciousness” is used here in a purely operational sense, referring not to subjective identification or shared norms, but to the analytical assumption that individuals within the group face a comparable opportunity set and can therefore be treated as mutually comparable for the purpose of welfare evaluation. A type refers to a further subset within each group consisting of individuals who share similar propensity, patterns, or habits for taking certain action. This paper assumes that individuals within the same type are cardinally comparable. Within a group, we assume individuals are either cardinally unit-comparable or ordinally level-comparable.

Fourth, these assumptions on theoretical measurability and comparability within type/group are also in accord with our empirical procedures. When aggregating individual-level data by type, we directly employ the ATE estimator, which means, for individuals within the same type, we assume it is valid to apply a linear parametric model with regressing the outcome on certain action as the independent variable. On the other, when aggregating type-specific values by group to construct the capability of individuals who are belonging to

⁵ Here, the articles ‘an’ and ‘a’ are crucial. This paper treats identity in accord with Amartya Sen’s understanding “individual identity” as a kind of capability set (Sen, 1999b). Sen originally argues that an individual is conceived to be assigned multiple group identities and to be forming and re-forming their multiple identities (and/or their weights).

certain group, we consider various normative aggregation rules (Utilitarian, Rawlsian, etc.) should be applied. This is because, based on an imaginary position exchange, it is reasonable to assume that individuals with different types in the same group can regard each other’s outcomes as elements within their own capabilities ⁶.

Lastly, from a different view, the correspondence between an individual perspective and a group perspective is in accord with the correspondence between a freedom perspective and an equality perspective. The axioms that are necessary to argue the capability from the viewpoint of freedom required original formulation. We distinguish three dimensions of freedom, agency freedom, rational freedom and well-being freedom. These dimensions are in accord with what previous literature has discussed separately. As mentioned earlier, research on “equality of opportunity” justifies excluding individual responsible choices in measuring inequality from the perspective of freedom; namely, what Sen calls “agency freedom” (i.e., individuals having more options over which they hold decisiveness). This paper further requires that individuals have plural optimal alternatives, and also requires that “optimal” is reasonably well, according to the literature of “opportunity of freedom”. We call the former “the rational freedom” and the latter “the well-being freedom”. This paper therefore proposes an extended axiomatic structure of freedom that incorporates these three dimensions, and shows that the type-specific ATE estimator and aggregation rules of estimators are (and are not) satisfies with the axiomatic structure ⁷.

The remainder of this paper is organized as follows. Section 2 describes the context and the unique panel data used. Section 3 presents the econometric strategy, estimating the ATE of relevant behavior (going-out in this article) on functionings. Section 4 provides the theoretical core, examining properties of the ATE estimator in terms of normative axioms of freedom. Section 5 reports type-by-type estimators in data. Section 6 theoretically discusses several ways of aggregating capabilities across types. Section 7 provides the actual measurement of size and unfreedom of capability sets of certain groups, and compares the measurements among several aggregation rules. Section 8 concludes.

2 Context and Data

2.1 Context setting: Welfare of going-out for elderly

One important aspect of daily life for the elderly is going out. Activities such as visiting friends and family, shopping, attending social events, or simply taking a walk are crucial for maintaining physical health, mental well-being, and social connections. However, the ability to go out can be influenced by various factors, including health conditions, mobility,

⁶ The question of which aggregation rule to adopt is, for real individuals, deeply connected to the formation of group consciousness—specifically, how they perceive the positions that other individuals actually occupy as their own capabilities (diverse ways of life that are worth valuing). For example, we observed the formation of such group consciousness within the atomic bomb survivors’ movement. Whether such group consciousness can be observed in groups investigated in this paper remains an intriguing research question for the future.

⁷ Transition from aggregations within an individual to aggregations across individuals allows us to apply the existing arguments on distributive justice. Not only the axioms related to social welfare functions but also the axioms that are used in measuring inequality and poverty can be employed (Foster et al., 1984; Foster and Shorrocks, 1991; Alkire and Foster, 2011; Andreassen and Di Tommaso, 2018; among others).

availability of transportation, and social support. Therefore, evaluating welfare related to going out requires considering both the opportunities and constraints, especially faced by the elderly.

To construct opportunity sets and evaluate welfare in relation to going out, it is essential to assess the situation when a person goes out and when the same person stays home instead. Since these two activities are logically incompatible within an individual, traditional measurements of welfare related to going out have been based solely on achievements during the act of going out. However, this information alone is insufficient to construct opportunity sets for the elderly.

This study focuses on a fundamental activity that encapsulates a dichotomy—staying home versus going out. By considering both scenarios, we aim to provide a more comprehensive evaluation of the welfare of the elderly, taking into account the opportunities available to them and the choices they make.

2.2 Survey structure and data collection

Recognizing that staying home and going out are two complementary facets of the same activity, this paper explains how the achievements of functionings in one setting correspond to those in the other. To address this, we constructed an individual panel survey that collects information on the actual achievements of functionings as well as the behavior the person engaged in.

The survey was conducted in a suburban city near Tokyo, Japan, targeting three distinct samples: a census of individuals over 70 years of age with disabilities (hereafter, category-WD), a census of individuals over 70 years of age who are public care recipients (hereafter, category-CR), and a random sample of individuals over 70 years of age without disabilities or public care reception (hereafter, category-GE). To simplify the data, individuals who simultaneously had disabilities and received public care were excluded.

A pilot survey was conducted in February 2020 using the sample from category-GE. After assessing the effectiveness of the questionnaire, the panel survey for the three samples was launched in July 2020 and continued until January 2024, with data collected every January, July, and October. For this article, we use data from twelve consecutive waves, spanning from July 2020 to January 2024. The response rates for the first wave varied across groups: 32.5% for category-WD, 45.5% for category-CR, and 70.0% for category-GE. After reviewing the variables, we retained 245, 190, and 959 observations for category-WD, CR, and GE, respectively.

2.3 Activity variables: Going-out/Staying-home

In every wave of survey, we ask the person whether he/she went out or stayed home on the specified date (usually within a few days from delivery of questionnaire). The survey asks the respondents to choose between “Yes” or “No” as the answer to the specific question: “On the date of T, did you go out?” When the respondents chose “No”, we regard as they stayed home on the day⁸.

⁸ In this survey, the definition of going out depends on the respondent. It may be going for a walk, shopping, going hospital, unexpected or expected. Nevertheless, we assume this person took concrete behavior and we

The average going-out ratios during the whole period are provided in Table 1: approximately 63.5% for category-CR, 72.9% for category-WD, and 81.1% for category-GE. Note that these figures are representative. According to national statistics, the average going-out ratio for individuals aged 60 years and older was 72.4% in July 2020⁹, while the survey data for category-GE reports 73.0% for wave 1, which was conducted in the same month.

Table 1: Summary Statistics (Pooled from Jul. 2020 to Jan. 2024; twelve waves)

	category-CR	category-WD	category-GE
number of observation	776	1,265	6,308
ratio of going-out	0.635	0.729	0.811
average index achieved functionings	0.559 (0.256)	0.624 (0.276)	0.712 (0.270)
average index of achieved functionings when go-out	0.593 (0.266)	0.656 (0.280)	0.728 (0.273)
average index of achieved functionings when stay-home	0.500 (0.227)	0.538 (0.244)	0.642 (0.244)

Notes: Calculation by authors. Means of each variable are shown. Standard deviations are in parentheses. Number of individuals in the sample is 190 for category-CR, 245 for category-WD, and 959 for category-GE.

2.4 Welfare variables: Achievements of functionings

The survey collects information about the actual behavior of respondents on a particular day in relation to ten specific functionings. A unique aspect of the survey is that the list of functionings is presented in exactly the same wording (in Japanese), regardless of whether the respondents chose “going-out” or “not going-out.” This structure allows the survey to gather data on how respondents perceive and report their achievements in these functionings, irrespective of their choice regarding going out on the specified day. Importantly, although the questions are identical, respondents are directed to different pages depending on whether they report going out or staying at home. This design makes it less apparent that the same set of questions is being asked, thereby minimizing potential bias or priming effects related to the framing of the survey¹⁰. The list of ten functionings is shown in Table 2. It is constructed based on four categories: ‘security and relaxation,’ ‘sense of gaining,’ ‘joy and sociality,’ and ‘comfort with oneself.’ The first category addresses fundamental aspects discussed in philosophy and psychology, particularly related to being human. The second category extends the concept of utility in economics. These two categories alone are insufficient to evaluate functionings comprehensively, so the list incorporates Aristotle’s concept of social existence as the third category, which goes beyond the economic utility. Additionally, the fourth category is included, based on the concept of social primary goods proposed by John Rawls.

intend to evaluate its result from the viewpoint of capability.

⁹ <https://www.mlit.go.jp/toshi/tosiko/content/001410803.xlsx>

¹⁰ Since the survey was aimed at elderly participants, it was administered using paper questionnaires to accommodate their preferences and accessibility needs.

Together, these categories aim to comprehensively capture various aspects of well-being and human experience¹¹.

Table 2: List of functionings: Question “Please tell us about your evaluation for each item.”

category	item
security and relaxation	I had the sense of security and relaxation on the whole
sense of gaining	I did not incur too much burden including financial costs
	more or less realized what I wanted to (goals achievement)
	I felt good and healthy about my body and mind
	I maintained a sense of control
joy and sociality	I had some stimulating new experience or perspective
	I had some unexpected encounter or discovery
	I refreshed myself, enjoyed myself, or smiled
	I talked and socialized with people
comfortability with oneself	I had the sense of being myself and being comfortable

Notes: Table 2 in Gotoh and Kambayashi (2023). For each item, respondents answer one of three choices; “applicable,” “somewhat applicable,” or “not applicable.”

Some may argue that these ten functionings are not suitable for evaluating functionings because they are subjective evaluations. However, they are closer to the concept of “awareness” in medicine (as opposed to “other awareness”) in the sense that they are qualitative evaluations of specific experiences. In any case, they are not variables that can be directly controlled by individuals. In this paper, we treat them as major outcome variables.

According to Amartya Sen, the list and weights of functionings are theoretically open and should be defined according to the research subject and context. In this paper, each of the ten functionings is weighted equally, yielding a scalar index between 0 and 1 for each respondent. Respondents are asked to assess each functioning using one of three choices: “applicable,” “somewhat applicable,” or “not applicable.” These responses are evaluated as 1, 0.5, and 0 respectively. Unless otherwise noted, the individual data in this paper refers to the ratio of going-out and staying-home instances obtained from the twelve panel waves, as well as the arithmetic average of the ten functionings achieved when the person goes out and stays home, respectively¹².

The summary statistics for the index of achievement of functionings are shown in Table 1. The average indexes are provided for each group: category-CR achieved 0.56 on average,

¹¹ Because the primary focus of this paper is the aggregation process among individuals, we refer the justification of the list to Gotoh and Kambayashi (2023).

¹² It may be worth noting that the arithmetic average allows full substitution between functionings. If the ten items are classified into four different categories, “gain” and “pleasure,” which consist of four items, would be given four times the weight. Other aggregation methods could also be considered. One method is to take the average for each of the four categories and add them together with equal weight. Another typical method is the geometric mean, which ensures that if any one of the functionings is zero, the result is zero. Alternatively, weights reported by the individual could be adopted.

category-WD achieved 0.62, and category-GE achieved 0.71. The average levels of achievement of functionings appear to be highest for the general elderly category and lowest for public care users, with people with disabilities falling in the middle. The table also highlights an interesting observation: there is an apparent asymmetry between the indexes when someone goes out versus when someone stays home. On average, for every category, the index is higher when people go out than when they stay home. However, the deviation between the two behaviors is not constant across categories. The absolute value of the deviation is largest in category-WD at 0.12, while category-CR and category-GE have deviations of 0.09. The next step is to statistically confirm these observed patterns.

3 Average Treatment Effects of Going-out on Achieved Functionings

To evaluate the difference in welfare between when going-out and when staying-home, we apply the fixed effects model. According to Rubin's causal model, the causal effects of a treatment is measured by the difference between the average of individual treated outcomes and those untreated; namely, the average treatment effects (hereafter, ATE). When the data satisfies the randomness of assignment of treatment, no omitted variable bias, and the full-support, we can estimate ATE from observed data.

In our context, given the population as N , we assume that identification assumptions are satisfied and the ATE of going-out on achieved functionings are measured by the following fixed-effects estimation

$$F_{it} = \beta \cdot OUT_{it} + u_i + wave_t + \alpha + \varepsilon_{it}. \quad (1)$$

Here, F_{it} is the scalar index of achieved functionings for individual i ($\in [1, \dots, N] \equiv \mathbb{I}$) at wave t ($\in [1, \dots, T] \equiv \mathbb{T}$). For simplicity, we assume a complete panel structure on data which means T is common for all of members. Then, OUT_{it} is defined as the dummy variable that takes 1 if individual i goes out at wave t , and takes 0 if individual i does not go out at wave t . u_i represents the individual-specific fixed effects for individual i , and $wave_t$ shows the wave-specific effects for wave t . α indicates the type-specific components, and ε_{it} is the random error term. Ordinally Least Square consistently estimates β as ATE of going-out on achieved functionings. Here, β and α in Eq. (1) can be respectively calculated as

$$\hat{\beta} = \frac{\sum_{i,t} \left[\left\{ (F_{it} - \overline{F_{it|t}}) - \left(\overline{F_{it} - \overline{F_{it|t}}} \right)_{i,t} \right\} \left\{ (OUT_{it} - \overline{OUT_{it|t}}) - \left(\overline{OUT_{it} - \overline{OUT_{it|t}}} \right)_{i,t} \right\} \right]}{\sum_{i,t} \left[\left\{ (OUT_{it} - \overline{OUT_{it|t}}) - \left(\overline{OUT_{it} - \overline{OUT_{it|t}}} \right)_{i,t} \right\}^2 \right]}, \quad (2)$$

and

$$\hat{\alpha} = \tilde{F} - \hat{\beta} \cdot \tilde{P}, \quad (3)$$

where

$$\overline{OUT_{it}|_t} = \frac{1}{T} \sum_t OUT_{it} = \frac{T_i^O}{T} \equiv P_i$$

(proportion of going out of individual i)

$$\overline{F_{it}|_t} = \frac{1}{T} \sum_t F_{it} \equiv F_i$$

(average functionings of individual i)

$$\overline{\overline{OUT_{it}|_t}|_{i,t}} = \frac{1}{T \times N_p} \sum_{i,t} OUT_{it} \equiv \tilde{P}$$

(overall proportion of going out of individuals whose going-out ratio is P_i)

$$\overline{\overline{F_{it}|_t}|_{i,t}} = \frac{1}{T \times N_p} \sum_{i,t} F_{it} \equiv \tilde{F}$$

(overall average functionings of individuals whose going-out ratio is P_i).

By using F_i , P_i , $\tilde{P}$, and $\tilde{F}$, Eq. (2) and Eq. (3) can be rewritten as

$$\hat{\beta} = \frac{\sum_{i,t} [(F_{it} - F_i) (OUT_{it} - P_i)]}{\sum_{i,t} [(OUT_{it} - P_i)^2]} = \frac{\sum_i P_i (F_i^O - F_i)}{\sum_i P_i (1 - P_i)}, \quad (4)$$

and

$$\hat{\alpha} = \tilde{F} - \left(\frac{\sum_i P_i (F_i^O - F_i)}{\sum_i P_i (1 - P_i)} \right) \tilde{P} = \tilde{F} - \sum_i \left\{ \left(\frac{P_i}{CV_p} \right) (F_i^O - F_i) \right\}. \quad (5)$$

Because $\tilde{P}$, $\tilde{F}$, and $\sum_i P_i (1 - P_i)$ are constant within individuals whose going-out ratios are the same as P_i , Eq. (4) indicates that the ATE of these individuals corresponds to the weighted average of individual-level differences in achieved functionings between the going-out (F_i^O) and the average (F_i). The ATE is zero when each individual's achieved functionings are identical across both states. In this case, Eq. (5) simplifies to the overall average of achieved functionings in the individuals who has the same going-out ratio ($\tilde{F}$).

Conversely, if all members attain higher achieved functionings when going out ($F_i^O > F_i$), the ATE becomes positive ($\hat{\beta} > 0$), reflecting a welfare gain attributable to going-out behavior. However, Eq. (5) also shows that the larger this gain ($F_i^O - F_i \uparrow$), the more it must be discounted from the overall achieved functionings ($\hat{\alpha} \downarrow$). This discounting effect depends on the going-out ratio: the more frequently an individual engages in going out the greater the necessary adjustment to the ATE of going-out behavior.

Next question is why Eq. (4) or Eq. (5) is appropriate to measure the welfare for the individuals whose going-out ratio is the same. To examine the normative properties of Eq. (4) and Eq. (5), we introduce the definition of capability set.

4 Normative Properties of ATE

4.1 Definition of capability set and capability evaluation rule

(Basic Model)

Let us assume that there are $n \geq 2$ individuals that make up a society, $N = \{1, \dots, n\}$, $m \geq 2$ menu of functionings and an action set X . Given society N , there exists a set of group characteristics J with generic element j . For each $j \in J$ there is a unique subset N^j of N such that $0 \leq |N^j| \leq |N|^{13}$. N^j may be empty or identical to N for some $j \in J$. There exist $j \in J$ and $j' \in J$ such that $N^j \cap N^{j'} \neq \emptyset$. A group $j \in J$ is assumed to commonly have a certain social or biological constraints, then assumed to share a possible set of resource bundle Z_j and a possible set of a utilization ability K_j .

Each individual $i \in N^j$ has a utilization ability $\mathbf{c}_i \in K_j : Z_j \rightarrow \mathbb{R}_+^m$, which transforms a resource bundle $z_i \in Z_j$ to a m-tuple functionings $\mathbf{F}_i \in \mathbb{R}_+^m$. By taking an action $x \in X$ with his resource bundle and utilization ability, an individual $i \in N^j$ produces a m-tuple functionings $\mathbf{F}_i^x = (F_{i1}^x, \dots, F_{im}^x) \in \mathbb{R}_+^m$, where $F_{if}^x = c_{if}^x(z_i)$, $f \in \{1, \dots, m\}$. As a convention, we assume that $c_i^x(0, \dots, 0) = (0, \dots, 0)$.

Note that the menu of functionings and the set of actions are assumed here to be common across all individuals. When an individual chooses an action out of this set, it is transformed by his own resources and utilization ability, resulting in a m-tuple non-negative functionings¹⁴. With this basic model, let us define the concepts of dual action and type, which play a central role in this paper.

(Dual action)

A *dual action* is defined as a pair of actions $\{O, S\} = A \in X$, in which if O happens, S cannot happen and vice versa, and each of which independently brings about non-negative functioning value for an individual i , i.e., $\mathbf{F}_i^O \in \mathbb{R}_+^m$ and $\mathbf{F}_i^S \in \mathbb{R}_+^m$. Typical examples of a dual action include “going-out (O)” and “staying-home (S).” In the following, for simplicity, we assume an individual’s achieved functionings for each action to be scalar, by aggregating from 1 to m with a certain weight. That is $F_i^O \in \mathbb{R}_+$, $F_i^S \in \mathbb{R}_+$.

(Type)

A *type* is represented by a probabilistic mixture of two components of a dual action and is denoted by $p \in [0, 1]$, where $p = 1$ implies one component of a dual action is surely chosen and $p = 0$ implies the opposite component is surely chosen. The population of type $p \in [0, 1]$ is a subset of individuals belonging to group $j \in J$, which is simply indexed by $i \in N^p (= \{1, \dots, n^p\})$ ¹⁵. The achieved functionings vector of an individual i with a type p is described as $\mathbf{F}_{ip} = (F_{ip}^O, F_{ip}^S) \in \mathbb{R}_+^2$. The profile of achieved functionings vectors of all individuals with a type p is described as $\{\mathbf{F}_{ip}\}_{i \in N^p} \in \mathbb{R}_+^{2n^p}$, and the profile of achieved functionings vectors of all individuals with a group is described as $\{\mathbf{F}_{ip}\}_{i \in N^j, p \in [0, 1]} \in \mathbb{R}_+^{2n^j} \times [0, 1]$.

¹³ Let $N \setminus \bigcup_{g \in G} N^g$ represents the set of non-group elements in N .

¹⁴ Note that we leave open the question of whether an individual’s resources and utilization ability are a matter of individual choice, a social operating variable or others. However, when evaluating an individual’s capability, as discussed later, we implicitly assume changes in his/her utilization ability or resources within a certain range, which result in changes of m-tuple functionings $\mathbf{F}_i \in \mathbb{R}_+^m$.

¹⁵ More precisely, the population of type $p \in [0, 1]$ belonging to group j is denoted as $i \in N^p \cap N^j$. Yet, since the following discussion proceeds with a group as given, we use this simple notation.

Hereinafter, unless otherwise specified, $\{\mathbf{F}_{ip}\}_{i \in N^p} \in \mathbb{R}_+^{2n^p}$, $\{\mathbf{F}_{ip}\}_{i \in N^j, p \in [0,1]} \in \mathbb{R}_+^{2n^j} \times [0,1]$ are denoted by $\{\mathbf{F}_{ip}\}^p, \{\mathbf{F}_{ip}\}^j$ and the profile of achieved functionings vectors of all individuals in society $\{\mathbf{F}_{ip}\} \in \mathbb{R}_+^{2n} \times [0,1]$ is simply denoted by $\{\mathbf{F}_{ip}\}$.

Now we can define an individual's capability set and its evaluation functions. Given a possible set of resource bundles, Z_j , a possible set of a utilization abilities, K_j and a possible set of types, $p \in [0,1]$, an individual's capability is defined as follows.

(Definition) *capability set*:

The capability of individual $i \in N^j, C_i$, is defined as $C_i = C(Z_j, K_j; p) = \{(F_{ip}^O, F_{ip}^S) : F_{ip}^O = c_{ip}^O(z_i), F_{ip}^S = c_{ip}^S(z_i), \text{ for some } z_i \in Z_j, \mathbf{c}_i \in K_j \text{ and } p \in [0,1]\}$.

Note that, since Z_j, K_j and $p \in [0,1]$ are identical for each group, individuals belonging to the same group, albeit of different types, are assumed to have the same capability set¹⁶. We assume that the individual capability $C_i \subseteq \mathbb{R}_+^2$ satisfies the following three criteria.

- (i) Compactness: C_i is closed and bounded,
- (ii) Star-shapedness: For all $\mathbf{F}_{ip} \in C_i$ and $\sigma \in [0,1]$, $\sigma \mathbf{F}_{ip} \in C_i$.
- (iii) Comprehensiveness: For all $\mathbf{F}_{ip} \in C_i$, if $\mathbf{0} \leq \mathbf{F}'_{ip} \leq \mathbf{F}_{ip}$, $\mathbf{F}'_{ip} \in C_i$.

Let $\mathfrak{C}_i$ to be the set of conceivable capability sets of an individual $i \in N^j$, elements are $C_i, C'_i, C''_i, \dots \in \mathfrak{C}_i$, that satisfy (i), (ii) and (iii). Let $\mathfrak{C} = \bigcup \mathfrak{C}_i$. Let $\succsim$ be a binary relation over $\mathfrak{C}$, such that “ C'_i is at least as great as C_i ,” which satisfies reflexivity and transitivity. Let $C'_i \succ C_i \leftrightarrow [C'_i \succsim C_i \text{ not } (C_j \succsim C'_j)]$ and $C'_i \sim C_i \leftrightarrow [C'_i \succsim C_i \& C_i \succsim C'_i]$.

The “individual capability” formulated as an opportunity set with the concepts of type and group contains a wealth of information. In the following sections, we condense this information into simple numerical values by applying it to actual data, while preserving the basic structure of this formulation. More concretely, we explore the process of aggregating the achieved functionings of individuals belonging to different types and the same group into numerical representations of the size and shape of an individual capability.

This process raises questions such as: Who (or which type) should be compared with whom (or which type)? Who (or which type) should be the focus? And what weight should be assigned to each (or which type) when aggregating the data?

To address these questions, we examine two procedures of aggregation. One is the aggregation procedure for individual data within the same type, and the other is the aggregation procedure for type-specific data within the same group.

These procedures of aggregating individual-specific information within a type and aggregating type-specific information within a group are termed here the “Capability Evaluation Rule” (CER) defined as follows.

(Definition) *capability evaluation rule* (CER)

¹⁶ In empirical studies, a set of resource bundle Z_j and a set of a utilization ability K_j are interpreted as the actual measured profile of each individual in the group. That is, $\mathbf{z} \in \prod_{i \in N^j} Z_{ij}$, $\mathbf{c} \in \prod_{i \in N^j} K_{ij}$. In the following discussion, we assume that a change in $\mathbf{z}$ or $\mathbf{c}$ leads to a change in $\{\mathbf{F}_{ip}\}$, and a change in C_i .

A *capability evaluation rule (CER)* is defined as a mapping, which associates each profile of achieved functionings $\{\mathbf{F}_{ip}\}^j$ to the reflexive and transitive relation $\succsim$ on $\mathfrak{C}$.

In other words, a CER is expressed as a rule for evaluating whether one capability is better than, or equally good as, another capability corresponding to two individual data.

Specifically, multiple CERs can be conceived depending on the criteria (called axioms) for determining informational focus or informational space to assume. Here, the informational space refers to differences such as cardinality (supporting arithmetic operations), ordinality (allowing ranking but not arithmetic operations), whether units (scales) are comparable across individuals, or whether levels are comparable across individuals.

With this basic model and key concepts, next, we define some notations which are crucial to identify the capability as freedom.

4.2 Social and Individual Evaluations and Informational Spaces

(Definition) *binary relation on the set of dual actions and on the set of types*
For any $O, S \in A$, for any $p \in [0, 1]$, OQ^pS represents a binary relation on the set of dual actions, such that “action O is at least as good as action S .” Let $O\overleftarrow{Q}^pS \leftrightarrow [OQ^pS \& \text{not}(SQ^pO)]$ and $O\overleftrightarrow{Q}^pS \leftrightarrow [OQ^pS \& SQ^pO]$.

For any $p, q \in [0, 1]$, pVq represents a binary relation on the set of types, defined on $[0, 1]$, such that “type p is at least as good as type q .” Let $p\overleftarrow{V}q \leftrightarrow [pVq \& \text{not}(pVq)]$ and $p\overleftrightarrow{V}q \leftrightarrow [pVq \& pVq]$.

We assume that both of binary relations Q^p and V satisfy reflexivity and transitivity but not necessarily satisfies completeness. Q^p and V denote social evaluations defined on the set of dual actions in type p and on the set of types in a group, respectively. By adding a subscript i to Q^p and V , we can define individual evaluations Q_i^p and V_i , respectively. Here, one conceivable method to construct social evaluations Q^p and V is by aggregating the individual evaluations, $\{Q_i^p\}$ and $\{V_i\}$. This paper does not inquire into this kind of consideration. The issue here is to explore the normative properties of Q and V , assuming that individuals in the same group can form and accept social evaluations based on a common group consciousness. Let $\mathfrak{Q}$ be conceivable set of binary relations on dual actions, elements are $Q^p, Q'^p \dots \in \mathfrak{Q}$ and $\mathfrak{V}$ be conceivable set of binary relations on types, elements are $V, V', \dots \in \mathfrak{V}$.

As mentioned above, since V (as well as Q) does not satisfy completeness, the existence of an optimal set (a set of elements that are at least as good as any other elements) is not guaranteed. Instead, we define a “maximal set,” as follows.

(Definition) *maximal value and maximal set:*

An action $x_{Q^p}^{\max} \in A$ is called a **maximal value** with respect to Q^p if and only if $\left[\nexists x' \in A, x'\overleftarrow{Q}^p x_{Q^p}^{\max} \right]$.

The set of maximal values in A is denoted by $Max(A, Q^p)$ and is called a **maximal set**.¹⁷
 A type $p_V^{\max} \in [0, 1]$ is called a **maximal value** with respect to V if and only if $\left[\nexists q \in [0, 1], q \overleftarrow{V} p_V^{\max} \right]$.
 The set of maximal values in $[0, 1]$ is denoted $Max([0, 1], V)$ and is called a **maximal set**.

It is known that the maximal set is not an empty set if V satisfies the acyclicity and the set is finite.¹⁸ Similarly, we define a “minimal set”.

(Definition) minimal value and minimal set:

An action $x_{Q^p}^{\min} \in A$ is called a **minimal value** with respect to Q^p if and only if $\left[\nexists x' \in A, x_{Q^p}^{\min} \overleftarrow{Q^p} x' \right]$.
 The set of minimal values in A is denoted by $Min(A, Q^p)$ and is called a **minimal set**.
 A type $p_V^{\min} \in [0, 1]$ is called a **minimal value** with respect to V if and only if $\left[\nexists p' \in [0, 1], p_V^{\min} \overleftarrow{V} p' \right]$.
 The set of minimal values in $[0, 1]$ is denoted by $Min([0, 1], V)$ and is called a **minimal set**.

In words, $Max([0, 1], V)$ denotes the set of elements “which are not dominated by any of the others” (elements in a set are equivalent or incomparable to one another), while $Min([0, 1], V)$ denotes the set of elements “which are not dominating any of the others” (elements in a set are equivalent or incomparable to one another).

Based on the definition of capability set and CER above, hereinafter, we concentrate our discussion on the empirical process of aggregating individuals’ actual data to construct the capability set. In the empirical study, the informational bases of the social evaluation can be obtained in real number (i.e. functionings values in this context). Therefore, we prepare for converting the binary relations defined above into the comparability on real number. However, there is one issue to consider.

Within an individual i , functionings that are brought about by different actions, at different times and with different probabilities can be assumed to be cardinally comparable to each other. However, it is non-trivial to what extent functionings are inter-personally measurable and comparable, since their origins or scales might be different depending on groups, types or individuals.

Below, we formally list several informational bases that differ in terms of measurability and comparability, for simplicity, with an assumption of an individual’s achieved functionings to be scalars, $F_{ip} \in \mathbb{R}_+$, by aggregating $F_{ip}^O \in \mathbb{R}_+$ and $F_{ip}^S \in \mathbb{R}_+$ with a certain weight.

(Definitions) ordinal non-comparability:

For some ψ_i , for any $i \in N^j$, for any $p, p' \in [0, 1]$, for any $F_{ip'}, F_{ip} \in \mathbb{R}_+$, $p' V p \leftarrow F_{ip'} \geq \psi_i(F_{ip})$ and for all $k \neq i \in N^j$, $F_{kp'} = \psi_k(F_{kp})$.

¹⁷ In this case, there are only three possible solutions. $Max(A, Q^p) = \{O\}, \{S\}$ or $\{O, S\}$.

¹⁸ Note that we can define a maximal set with the assumption of inter-personal non-comparability. In this case, every individual’s optimal with respect to V_i should be left in the maximal set as being mutually non-dominated. Or only randomly selected optimal values will remain.

For some ψ_i , for any $i \in N^j$, for any $p \in [0, 1]$, for any $x, x' \in A$, for any $F_{ip}^{x'}, F_{ip}^x \in \mathbb{R}_+$, $x'Q^px \leftarrow F_{ip}^{x'} \geq \psi_i(F_{ip}^x)$ and for all $k \neq i \in N^j$, $F_{kp'} = \psi_k(F_{kp})$.

(Definitions) ordinal inter-personal comparability:

For some φ , for all $i \in N^j$, for any $p, p' \in [0, 1]$, for any $F_{ip'}, F_{ip} \in \mathbb{R}_+$ $p'Vp \leftarrow F_{ip'} \geq \varphi(F_{ip})$. and for all $k \neq i \in N^j$, $F_{kp'} = \varphi(F_{kp})$.

For some φ , for all $i \in N^j$, for any $p \in [0, 1]$, for any $x, x' \in A$, for any $F_{ip}^{x'}, F_{ip}^x \in \mathbb{R}_+$, $x'Q^px \leftarrow F_{ip}^{x'} \geq \varphi(F_{ip}^x)$. and for all $k \neq i \in N^j$, $F_{kp'} = \varphi(F_{kp})$.

(Definitions) unit inter-personal comparability:

For some a_i, μ , for all $i \in N^j$, for any $p, p' \in [0, 1]$, for any $F_{ip'}, F_{ip} \in \mathbb{R}_+$ $p'Vp \leftarrow F_{ip'} \geq a_i + \mu F_{ip}$. and for all $k \neq i \in N^j$, $F_{ik} = a_k + \mu F_{kp}$.

For some a, ρ , For all $i \in N^j$, for any $p \in [0, 1]$, for any $x, x' \in A$, for any $F_{ip}^{x'}, F_{ip}^x \in \mathbb{R}_+$, $x'Q^px \leftarrow F_{ip}^{x'} \geq a_i + \rho F_{ip}^x$. and for all $k \neq i \in N^j$, $F_{kp'} = a_k + \mu F_{kp}$

(Definitions) inter-personal cardinal (full) comparability:

For some a, ρ , for all $i \in N^j$, for any $p, p' \in [0, 1]$, for any $F_{ip'}, F_{ip} \in \mathbb{R}_+$, $p'Vp \leftarrow F_{ip'} \geq a + \rho F_{ip}$. and for all $k \neq i \in N^j$, $F_{kp'} = a + \rho F_{kp}$.

For some a, ρ , For all $i \in N^j$, for any $p \in [0, 1]$, for any $x, x' \in A$, for any $F_{ip}^{x'}, F_{ip}^x \in \mathbb{R}_+$, $x'Q^px \leftarrow F_{ip}^{x'} \geq a + \rho F_{ip}^x$. and for all $k \neq i \in N^j$, $F_{kp'} = a + \rho F_{kp}$.¹⁹

In this paper, (i) we assume full cardinal inter-personal comparability within the same type in the same group, (ii) while only ordinal level inter-personal comparability or cardinal unit inter-personal comparability between different types in the same group. (iii) Across different groups, we assume only incomplete ordinal level inter-personal comparability (for example, comparability on certain numerical representations of individual capabilities).

Let us denote the set of conceivable profiles, $\{F_{ip}\} \in \mathbb{R}_+^{n^j}$, that satisfies ordinal level inter-personal comparability by $\bar{L}$, cardinal unit inter-personal comparability by L_u , and full cardinal inter-personal comparability by L . In addition, we denote ordinal inter-personal non-comparability by L_{non} .

4.3 Definition of freedom

Next, to examine the normative properties of CER, we define several concepts related to freedom with different informational bases of measurability-comparability. First, with the ordinal inter-personal non-comparability L_{non} , we can define decisiveness and agency freedom axiom.

¹⁹ Note that an invariance of functioning values is a sufficient condition for equal evaluation V , but not a necessary one. For two actions or types with the same values, for example, additional conditions may be applied to determine that one is strictly better than the other.

(Definition) decisiveness for actions and types

For any individual $i \in N$, for any $p \in [0, 1]$, for any pair of alternatives $O, S \in A$, if $O \overleftarrow{Q}_i^p S \rightarrow$ not $S \in \text{Max}(A, Q^p)$ and $S \overleftarrow{Q}_i^p O \rightarrow$ not $O \in \text{Max}(A, Q^p)$ hold, the individual is decisive in this society over a pair of $O, S \in A$.

For any individual $i \in N$, for any pair of alternatives $p, p' \in [0, 1]$, if $p \overleftarrow{V}_i p' \rightarrow$ not $p' \in \text{Max}([0, 1], V)$ and $p' \overleftarrow{V}_i p \rightarrow$ not $p \in \text{Max}([0, 1], V)$ hold, the individual is decisive in this society over a pair of $p, p' \in [0, 1]$.

In words, if an alternative O (resp. S) is strictly preferred to alternative S (resp. O) by a person, then S (resp. O) must not be socially chosen if O (resp. S) is available. Similarly, if an alternative p (resp. p') is strictly preferred to alternative p' (resp. p) by a person, then p' (resp. p) must not be socially chosen if p (resp. p') is available. Recall that the definition of “minimum liberalism” by Sen (Sen, 1970) in the context of so-called “liberal paradox” is that there are at least two persons and two pairs of distinct alternatives such that each individual is decisive over each pair in either order (bidirectionally).

(Definition) agency freedom axiom:

Let us denote the number of pairs $O, S \in A$, for which an individual i 's decisiveness holds, by $|D(A, Q_i^p, Q^p)|$, and denote the range of pairs $p, q' \in [0, 1]$, for which an individual i 's decisiveness holds, by $|D([0, 1], V_i, V)|$, then, agency freedom is defined as the following statement.

(i) For any individual $i \in N$, for any $C_i, C'_i \in \mathfrak{C}$, for any $p \in [0, 1]$, and for any $Q_i^p, Q^p \in \mathfrak{Q}$ corresponding to C_i , and for any $(Q'_i)^p, (Q')^p \in \mathfrak{Q}$ corresponding to C'_i , C'_i is at least as free as C_i if $D(A, (Q'_i)^p, (Q')^p) \geq D(A, Q_i^p, Q^p)$, and for all $q \neq p$ with $q \in [0, 1]$, not $D(A, (Q'_i)^q, (Q')^q) < D(A, Q_i^q, Q^q)$.

(ii) For any individual $i \in N$, for any $C_i, C'_i \in \mathfrak{C}$, for any $p \in [0, 1]$, for any $V_i, V \in \mathfrak{V}$ corresponding to C_i , and for any $V'_i, V' \in \mathfrak{V}$ corresponding to C'_i , C'_i is at least as free as C_i if $D([0, 1], V'_i, V') \geq D([0, 1], V_i, V)$.

In words, agency freedom axiom states that if the *number* of pairs $O, S \in A$, for which the decisiveness holds, is greater, freedom in capability is greater. Besides, if the *range* of pairs $p, p' \in [0, 1]$, for which the decisiveness holds, is greater, freedom in capability is greater. For an individual i of type p , an increase in the value of type p , provided that no other types object, is regarded as an improvement in i 's freedom.

We admit that agency freedom axiom defined bi-directionally is crucial. If a person's choice for doing an action (i.e. going out) is respected, but her choice for not doing the action (i.e. not going out) is ignored, or vice versa, it is difficult to say that freedom of choice is guaranteed. Note that this definition of freedom is more substantive than so-called “negative freedom (absence of external intervention)” in the sense that it requires the actions chosen by the individual to be actually realized. Yet it has the following difficulties in measurement.

First, it seems difficult to directly measure agency freedom because it is defined based

on the individual's subjective evaluation²⁰. In revealed preference theory, for example, the chosen action is interpreted as the one that the individual prefer to. Yet, in order to say so, it is necessary to verify whether it was possible to choose the opposite direction if one had intended to do so. Second, it is necessary to look at not only the value of chosen action but also the value of unchosen actions. Suppose an individual achieves a high value of functionings with a high going-out ratio, but achieves only a miserable value of functionings with a low going-out ratio. Then, can we say he is free in his choice?

Therefore, there is a reason for focusing on the opportunity set itself, and we provide an additional concept of freedom, the rational freedom, by introducing the assumption of ordinal inter-personal comparability²¹.

(Definition) rational freedom axiom:

i) Let us denote the number of maximal values in a maximal set by $|Max([0, 1], V)|$ and $|Max(A, Q)|$. The **rational freedom** requires that if the number of maximal values is greater, an individual freedom is greater. It is formally written as follows.

i-1) For any $C_i', C_i \in \mathfrak{C}$, for any $V, V' \in \mathfrak{V}$ corresponding to C_i and C_i' respectively, C_i' is at least as free as C_i , if $|Max([0, 1], V')| \geq |Max([0, 1], V)|$.

i-2) For any $C_i', C_i \in \mathfrak{C}$, for any $p \in [0, 1]$, for any $Q^p, Q'^p \in \mathfrak{Q}$ corresponding to C_i and C_i' respectively, C_i' is at least as free as C_i , if $|Max(A, Q'^p)| \geq |Max(A, Q^p)|$

and for all $q \neq p \in [0, 1]$, not $|Max(A, Q'^q)| < |Max(A, Q^q)|$.

ii) Let us denote the number of minimal values in a minimal set by $|Min([0, 1], V)|$ or $|Min(A, Q^p)|$. The **rational freedom** requires that if the number of minimal values is greater, freedom is greater. It is formally written as follows.

ii-1) For any $C_i', C_i \in \mathfrak{C}$, and for any $V, V' \in \mathfrak{V}$ corresponding to C_i and C' , respectively, C_i' is at least as free as C_i , if $|Min([0, 1], V')| \geq |Min([0, 1], V)|$.

ii-2) For any $C_i', C_i \in \mathfrak{C}$, for any $p \in [0, 1]$, for any $Q^p, Q'^p \in \mathfrak{Q}$ corresponding to C_i and C_i' respectively, C_i' is at least as free as C_i , if $|Min([0, 1], Q'^p)| \geq |Min([0, 1], Q^p)|$

and for all $q \neq p \in [0, 1]$, not $|Max(A, Q'^q)| < |Max(A, Q^q)|$.

Obviously, when the maximal possible number of $|Max([0, 1], V)|$ includes all observable probabilistic actions between 0 and 1, and that of $|Max(A, Q^p)|$, these situations imply that all pairs of types or actions are mutually equal or incomparable with respect to V or Q^p . In this case, for example, $Max([0, 1], V)$ and $Min([0, 1], V)$ coincide. That is, no one is dominated by anyone, and no one dominates anyone.

²⁰ The following indirect methods can be considered: (1) If a person felt a sense of control when choosing a particular pair of options, that choice is considered decisive. (2) Cases where a person went out despite "wanted to avoid going out," or stayed home despite "wanted to avoid staying home," are considered non-decisive.

²¹ If we introduce the assumption of cardinal, but inter-personal non-comparability, the principle of expected utility maximization under uncertainty can be suitably applied. This method requires that an individual should aggregate the value of all possible alternatives with a certain weight.

The rational freedom captures an important aspect of the freedom emphasized by Sen. That is, “the freedom they have to choose between different ways of living they can have reason to value (Sen, 2017, 357).” This can be paraphrased as that if there is only “one way of living,” it may be a rational necessity, different from the rational freedom, to choose it. However, rational freedom has its own limitation. While it requires maximization of the number of the maximal set, it does not care about the decency of the maximal set. Extremely, the rational freedom allows the elements of the maximal set to be uniformly miserable. Therefore, we further provide an additional concept of freedom, the well-being freedom.

(Definition) well-being freedom axiom:

A rule satisfies the well-being freedom if either i) or ii) satisfies.

*i) Let us denote the level of maximal values by $|p_V^{\max}|$ and $|x_{Q^p}^{\max}|$. The **well-being freedom** requires that if the level of maximal values is higher, an individual freedom is greater. It is formally written as follows.*

i-1) For any $C_i', C_i \in \mathfrak{C}$, for any $V, V' \in \mathfrak{V}$ corresponding to C_i and C_i' respectively, C_i' is at least as free as C_i , if $|p_{V'}^{\max}| \geq |p_V^{\max}|$.

i-2) For any $C_i', C_i \in \mathfrak{C}$, for any $p \in [0, 1]$, for any $Q^p, Q'^p \in \mathfrak{Q}$ corresponding to C_i and C_i' respectively, C_i' is at least as free as C_i , if $|x_{Q'^p}^{\max}| \geq |x_{Q^p}^{\max}|$ and for all $q \neq p \in [0, 1]$, not $|x_{Q'^q}^{\max}| < |x_{Q^q}^{\max}|$.

*ii) Let us denote the level of minimal values by $|p_V^{\min}|$ and $|x_{Q^p}^{\min}|$. The **well-being freedom** requires that if the level of minimal values is higher, freedom is greater; formally written it is,*

ii-1) For any $C_i', C_i \in \mathfrak{C}$, for any $V, V' \in \mathfrak{V}$ corresponding to C_i and C' , respectively, C_i' is at least as free as C_i , if $|p_{V'}^{\min}| \geq |p_V^{\min}|$.

ii-2) For any $C_i', C_i \in \mathfrak{C}$, for any $p \in [0, 1]$, for any $Q^p, Q'^p \in \mathfrak{Q}$ corresponding to C_i and C_i' respectively, C_i' is at least as free as C_i , if $|x_{Q'^p}^{\min}| \geq |x_{Q^p}^{\min}|$ and for all $q \neq p \in [0, 1]$, not $|x_{Q'^q}^{\min}| < |x_{Q^q}^{\min}|$.

The background idea of the well-being freedom is the following: while the rational freedom requires maximization of the number of the maximal set, it does not care about the level (height) of the maximal elements. In contrast, the well-being freedom requires maximization of the level (height), taking advantage of the assumption of the ordinal level inter-personal comparability of indices.

4.4 Definition of size and shape of capability set, the well-being freedom, and ATE

In what follows, we examine capability evaluation rules (CER), the domain of which is restricted to L (full cardinal inter-personal comparability). Since actual data include the time dimension, our informational bases now consists of $\{\mathbf{F}_{ip}^t\}_{i \in N, p \in [0,1], t \in \{1, \dots, T\}} \in L$. Here, without loss of generality, we can rearrange the ordering of data so that the person of type p goes out from wave 1 to wave pT and stay home from wave $pT + 1$ to wave T . Then, the element of the informational bases of capability sets, $\mathbf{F}_{ip}$ is expressed as a two-dimensional vector representing the average of the achieved functioning of the person over time, separately aggregated for going-out and staying-home; namely,

$$\overline{\mathbf{F}}_{ip} = \left(\frac{1}{pT} \sum_{(t)^O=1}^{pT} F_{ip}^{O(t)^O}, \frac{1}{(1-p)T} \sum_{(t)^S=pT+1}^T F_{ip}^{S(t)^S} \right),$$

where $(t)^O \in \{1, \dots, pT\}$ and $(t)^S \in \{pT + 1, \dots, T\}$.

Below, we first define basic metrics for constructing the size and shape of capability set. Next, based on these metrics, we propose two capability evaluation rules that satisfy the axioms of efficiency and freedom. Finally, we make explicit the desirable properties of ATE through these capability evaluation rules.

Before starting discussions, let us first define the total and the difference of the achieved functioning for going-out and staying-home within a type.

(Definition) the total and the difference within a type:

For each $p \in [0, 1]$ the **total** of the achieved functionings for all $i \in N^p$ can be expressed as,

$$\begin{aligned} F^p &= \frac{1}{|N^p|} \sum_{i=1}^{n^p} \left(\frac{pT}{T} \frac{1}{pT} \sum_{(t)^O=1}^{pT} F_{ip}^{O(t)^O} + \frac{(1-p)T}{T} \frac{1}{(1-p)T} \sum_{(t)^S=pT+1}^T F_{ip}^{S(t)^S} \right) \\ &= \frac{1}{|N^p|T} \sum_{i=1}^{n^p} \left(\sum_{(t)^O=1}^{pT} F_{ip}^{O(t)^O} + \sum_{(t)^S=pT+1}^T F_{ip}^{S(t)^S} \right). \end{aligned}$$

While, the **difference** between the achieved functionings of $O \in A$ and $S \in A$ for all $i \in N^p$, can be expressed as,

$$\beta^p = \frac{1}{N^p T} \left| \sum_{(t)^O=1}^{pT} F_{ip}^{O(t)^O} - \sum_{(t)^S=pT+1}^T F_{ip}^{S(t)^S} \right|.$$

Next, let us define the concept of the *size* for dual actions, which takes into account both

the total and the difference between the achieved functionings of $O \in A$ and $S \in A$ for all $i \in N^p$.

(Definition) the *size combining the difference and the total*:

For each $p \in [0, 1]$, **size** α^p is defined as the function of the difference, β^p , and the total, F^p , that is, $\alpha^p = \varphi(F^p, \beta^p)$ where α^p is an increasing function of $F^p (\geq 0)$ and a decreasing function of $\beta^p (\geq 0)$ ²².

One typical examples is a family of linear combinations of the two such that: $\alpha^p = \zeta F^p - \eta \beta^p$, where $\zeta \geq 0$. Although this is an extremely simple formula, as the following two examples show, depending on the values of the two parameters, ζ and η , it can include min-max value as well as max-max value.

For example, taking $\zeta = \eta = 1$, the above equation holds as follows. $\alpha^p = F^p - \beta^p = (F_{ip}^O + F_{ip}^S) - |F_{ip}^O - F_{ip}^S|$. Then, we can confirm that α^p represents $x_{Q^p}^{\min}$, since if $F_{ip}^O > F_{ip}^S$, then $\alpha^p = 2F_{ip}^S$. Conversely, if $F_{ip}^O < F_{ip}^S$, then $\alpha^p = 2F_{ip}^O$. On the other hand, taking $\zeta = 1, \eta = -1$, the above equation holds as $\alpha^p = F^p + \beta^p = (F_{ip}^O + F_{ip}^S) + |F_{ip}^O - F_{ip}^S|$. Then, we can confirm that α^p represents $x_{Q^p}^{\max}$, since if $F_{ip}^O > F_{ip}^S$, then $\alpha^p = 2F_{ip}^O$. Conversely, if $F_{ip}^O < F_{ip}^S$, then $\alpha^p = 2F_{ip}^S$.

With concepts of the size, α^p , and the difference, β^p , let us define the following two capability evaluation rules within a type.

(Definition) α^p -maximization rule:

For any $C_i', C_i \in \mathfrak{C}$, for any $i \in N^p$, for any $p \in [0, 1]$, for any $\{\alpha'^p\}, \{\alpha^p\} \in \mathbf{L}$ corresponding to C_i', C_i , respectively, $C_i' \succsim C_i$, if and only if $\alpha'^p \geq \alpha^p$ and for all $q \neq p \in [0, 1]$, not $\alpha'^q < \alpha^q$.

(Definition) β^p -minimization rule:

For any $C_i', C_i \in \mathfrak{C}$, for any $i \in N^p$, for any $p \in [0, 1]$, for any $\{\beta'^p\}, \{\beta^p\} \in \mathbf{L}$ corresponding to C_i', C_i , respectively, $C_i' \succsim C_i$, if and only if $\beta'^p \leq \beta^p$ and for all $q \neq p \in [0, 1]$, not $\beta'^q < \beta^q$.

Hereafter, we attempt to examine the normative properties of these two rules by several axioms including the concepts of freedom. Before proceeding, let us define strong and weak Pareto conditions concerning two criteria, α^p and β^p .

(Definition) strong Pareto axiom for criteria:

For any $p \in [0, 1]$, for any $\{F^p\}, \{\beta^p\} \in L \cap \mathbb{R}_+$ corresponding to C_i , and $\{F'^p\}, \{\beta'^p\} \in$

²² Note that, by definition, both α^p and β^p are elements belonging to an individual's capability set.

$L \cap \mathbb{R}_+$ corresponding to C'_i , $C'_i \succ C_i$, if $\beta'^p \leq \beta^p$ and $F'^p \geq F^p$, either $\beta'^p < \beta^p$ or $F'^p > F^p$, and for all $q \neq p \in [0, 1]$, not $\beta'^q > \beta^q$ or $F'^q > F^q$.

In words, strong Pareto axiom for criteria requires that within a type p , if a change improves one of “difference, β^p ” and “total, F^p ” without decreasing either, moreover, for any other types $q \neq p \in [0, 1]$, both difference and total remains the same, such a change improves an individual’s capability. By weakening the conditions slightly, we can have the following axiom.

(Definition) weak Pareto axiom for criteria:

For any $p \in [0, 1]$, for any $\{F^p\}$, $\{\beta^p\} \in L \cap \mathbb{R}_+$ corresponding to C_i , and $F'^p, \{\beta'^p\} \in L \cap \mathbb{R}_+$ corresponding to C'_i , $C'_i \succ C_i$, if $\beta'^p < \beta^p$ and $F'^p > F^p$, and for all $q \neq p \in [0, 1]$, not $\beta'^q > \beta^q$ or $F'^q > F^q$.

Then we have the following lemma (proofs are in from Appendix B to Appendix D, respectively).

Lemma 1:

- (1) α^p -maximization rule satisfies the strong Pareto for plural criteria.
- (2) β^p -minimization rule does not satisfy the strong Pareto for plural criteria but satisfy the week Pareto for plural criteria.

Lemma 2:

α^p -maximization rule satisfies the well-being freedom but not satisfy the rational freedom.

Lemma 3:

- (1) For any $i \in \mathbb{N}^p$, for each $p \in [0, 1]$, for any $\{\beta'^p\}, \{\beta^p\} \in \mathbf{L}$ corresponding to C'_i, C_i , $|\text{Max}(\mathbf{A}, \mathbf{Q}'^p)| \geq |\text{Max}(\mathbf{A}, \mathbf{Q}^p)|$, if $\beta'^p \leq \beta^p$.
- (2) β^p -minimization rule satisfies rational freedom, but not satisfy well-being freedom.

This lemma shows that when the difference β^p becomes smaller, the number of elements of the maximal set becomes larger. Hence, by rational freedom, which requires that a larger maximal number corresponds to greater individual freedom, we can conclude that a decrease in the difference β^p implies an increase in individual freedom. In this sense, we call the

difference β^p “unfreedom” in this paper²³.

Our interest was to see what normative properties the empirical methods such as ATE satisfies. The following theorem provides one possible response to this question.

Proposition 1:

Here we apply ATE in Eq. (1) to the sample of type p (i.e. the individuals who has the same going-out ratio) and define estimated type-by-type coefficients as $\widehat{\alpha}^p$ and $\widehat{\beta}^p$, similarly to Eq. (2) and Eq. (3). Then, $\widehat{\alpha}^p$ –maximization rule satisfies the well-being freedom, while $\widehat{\beta}^p$ –minimization rule satisfies the rational freedom.

Proof: See Appendix E.

According to Proposition 1, $\widehat{\alpha}^p$ in Eq. (1) captures the size of capability associated with each type, while $\widehat{\beta}^p$ reflects the degree of unfreedom in capability. Thus, type-specific ATE provides a way to jointly assess both well-being freedom and rational freedom at the individual level.

Assessing individual capability requires evaluating personal freedom in relation to both sides of dual actions—such as going out versus staying home. To do so, one needs information on both the difference and sum of outcomes associated with these mutually exclusive actions, while controlling for confounding factors. However, such information cannot be directly observed, since an individual cannot simultaneously realize both actions at a given point in time.

To address this empirical limitation, the present study builds on the econometric implications of the ATE framework. Unlike cross-sectional comparisons of outcomes across individuals, the ATE estimator evaluates, for each individual, the average achieved functionings under alternative actions, and often produces different results from simple OLS. In this sense, Proposition 1 enables the evaluation of capability sets using within-individual variation in actions, a perspective that is central to the capability approach, which emphasizes counterfactual opportunities rather than merely realized outcomes. Moreover, as shown in the next section, the simple difference in achieved functionings between going-out and staying home in Table 1 is much larger than the results from fixed-effect estimator, and the researchers have struggled with which estimator is more appropriate to measure the capability. The proposition offers a tractable and familiar way to represent individual capability numerically through estimators, thereby empirically facilitating the assessment of both rational freedom and well-being freedom. In this way, rather than relying on simple OLS comparisons, the ATE framework enables capability to be measured on the basis of within-individual counterfactual variation, making it both normatively consistent with the capability approach and empirically appropriate. In the next section, we assess the empirical usefulness of this approach using actual data.

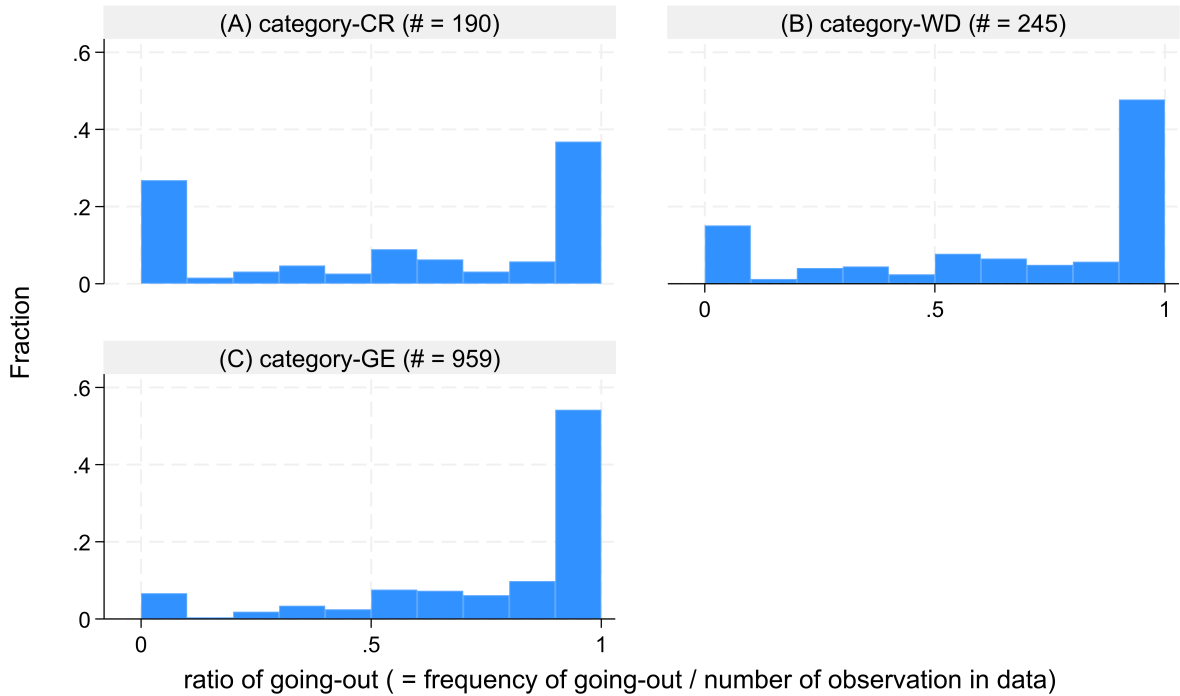
²³ For the usage of the word “unfreedom,” see, for example, the following: “economic unfreedom can breed social unfreedom, just as social or political unfreedom can also foster economic unfreedom” (Sen (1999a), 8)

5 Measurement of Size and Unfreedom of Capability

5.1 Empirical definition of type

Given Proposition 1, we can measure the actual size of capability and unfreedom for each type using the type-by-type estimators $\widehat{\alpha}^p$ and $\widehat{\beta}^p$. To begin, we strictly define the *type* within our dataset. Since our dataset is an unbalanced panel, a simple count of going-out instances does not uniquely identify a behavioral pattern; for example, a one-third going-out ratio could result from going out three times in nine waves, or once in three waves. Therefore, we define *type* explicitly by the ratio of going-out behavior (p).

Figure 1: Distribution of *type* in Data



Notes: Authors calculated. The unit of observation is person. Some observations are not used in the estimation because of lack of variables.

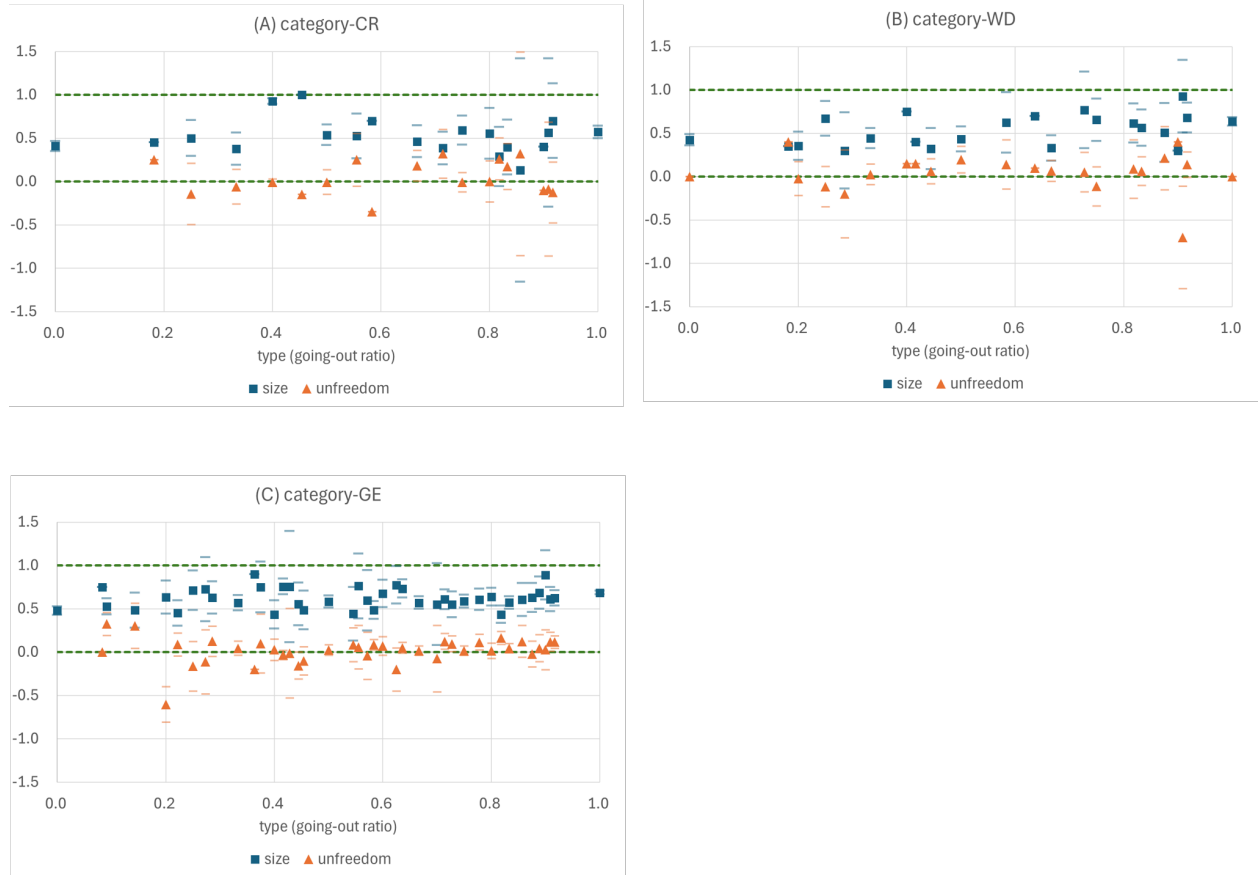
Figure 1 illustrates the distribution of these types across groups. Category-CR and Category-WD exhibit a polarized distribution, concentrating heavily on the extremes of zero (always-stay-home) or one (always-go-out). In contrast, Category-GE shows a concentration at the ratio of one, but also maintains a presence across intermediate ratios. While the discrete nature of these types results in varying sample sizes — sometimes leading to larger standard errors for rare types — this granularity is essential to capture the heterogeneity of

“ways of living.” Acknowledging this trade-off, we proceed to estimate the parameters for types with sufficient observations to ensure minimal statistical reliability.

5.2 Type-by-type aggregation

The estimated values for capability size ($\widehat{\alpha^p}$) and unfreedom ($\widehat{\beta^p}$) are presented in Figure 2 and Appendix A.

Figure 2: Graphical Presentation of size (α^p) and unfreedom (β^p)



Notes: Calculated by authors based on Appendix Table A. Horizontal line shows the going-out ratio of each type. Square and triangle are $\widehat{\alpha^p}$ and $\widehat{\beta^p}$ respectively, with lines for 95% confidence interval.

A striking finding is that neither the “always-go-out” nor the “always-stay-home” types exhibit the maximum capability size or the highest level of unfreedom. Instead, these extremes occur among individuals with moderate going-out frequency. For example, in Category-GE, the maximum capability size ($\widehat{\alpha^p} = 0.900$) is observed in Type-11 (ratio = 0.36), while the highest unfreedom ($\widehat{\beta^p} = -0.602$) appears in Type-5 (ratio = 0.20). This challenges

the conventional assumption that activity frequency directly determines welfare, suggesting instead that the ability to secure functionings both at home and outside is critical for capability and freedom²⁴.

The distribution of $\widehat{\beta^p}$ in Figure 2 illustrates the extent to which individuals are compelled to choose specific actions. In Category-GE, for instance, 33% of types (excluding two deterministic extremes) exhibit statistically significant $\widehat{\beta^p}$ values. For these individuals, the welfare gap between going out and staying home is substantial, implying a state of “unfreedom” where they are effectively constrained to select one option to avoid welfare loss.

When comparing categories, the maximum $\widehat{\alpha^p}$ within each group is similar across categories, suggesting that in any category, some individuals can achieve high functionings by adopting a specific going-out ratio. Regarding unfreedom ($\widehat{\beta^p}$), the proportion of types experiencing significant constraints shows no clear difference across categories: 39% in Category-CR, 35% in Category-WD, and 33% in Category-GE. This indicates that all categories face comparable limitations on freedom of choice.

However, these interpretations of comparison cannot serve as a foundation for rigorous normative assessment. Applying a simple α^p -maximization or β^p -minimization rule across categories fails to establish superiority between categories, as these rules assume that a type is preferable when transitioning between categories, provided it is not inferior for all other types in the same transition (in short, the condition of “without contradicting other types”). Let’s quickly check whether this condition is satisfied in Figure 2.

Comparing types with the α^p maximum among groups, the CR-group does indeed show superiority. However, comparing types with other values, from the second α^p maximum to the α^p minimum, the superiority among groups is not uniformly determined. Similarly, while the proportion of types with β^p minimum is not so much different among groups, types with other values shows significant differences among groups. Thus, the condition of “without contradicting other types” does not hold for either $\widehat{\alpha^p}$ or $\widehat{\beta^p}$, rendering such judgments inconclusive.

In summary, it follows that: Type-by-type estimation offers valuable empirical evidence on how well-being related to going out versus staying home is shaped. However, comparing capability across groups is not straightforward. To construct a comparable measure of group capability that accounts for both average experiences and minority realities among members, we must move beyond individual types. We therefore adopt across-type aggregation methods to integrate these diverse experiences into a unified social evaluation.

6 Aggregation across types

6.1 Capability evaluation rules regarding equality

In Section 4, we propose three concepts of freedom (agency freedom, rational freedom, and well-being freedom) and two capability evaluation rules within a type (the β^p -minimization rule and α^p -maximization rule) and examine the relationship between the concepts of free-

²⁴ This inference can be confirmed by an absence of correlation between capability size and unfreedom. The correlation coefficients between $\widehat{\alpha^p}$ and $\widehat{\beta^p}$ are negligible: -0.09 for Category-GE, -0.37 for Category-CR, and 0.20 for Category-WD.

dom and rules. Then we have an indication that β^p and α^p are appropriate as type-based information for estimating the size and the shape of the capability set as freedom. The listed $\widehat{\beta^p}, \widehat{\alpha^p}$ across all types shown in Figure 2 can be interpreted precisely as the “individual capability” in each group.

However, as mentioned above, the method for aggregating type-based information into the size and shape of an individual’s capability is not self-evident. The questions arise: when contradictions arise between types over advantages, where should the informational focus be placed for priority assignment—the maximum value, the minimum value, or the average? Or, should the number of people in each type be considered?

In this section, using the type-by-type estimates, $\widehat{\beta^p}, \widehat{\alpha^p}$ ($= \widetilde{F} - \widehat{\beta^p} \cdot \widetilde{P}$) to be type-based information, we investigate five CERs, including ones which are familiar in the context of inequality and justice with the assumption of cardinal unit comparability or ordinal level comparability across types. Advancing the technical discussion, let us briefly make a philosophical remark.

As noted earlier, the evaluation of one’s capability should involve equal concern and respect for others in a group, who have different “ways of living,” but whose positions are imaginatively interchangeable. This perspective opens up the possibility of capturing the overlapping relations between the concept of freedom and the concept of equality, thereby linking capability evaluation rules to broader principles of justice.

Nevertheless, a familiar philosophical conundrum remains here: what is the relevant range of such “others”? This paper addresses this question in an operational manner by linking it to informational space, on which capability evaluation rules are defined—specifically, two regimes of inter-personal comparability: cardinal versus ordinal, and within-type versus across-types.

First, with informational profiles, which are cardinally *unit* comparable, such as $\{\alpha^p, \alpha^q, \dots\}, \{\alpha'^p, \alpha'^q, \dots\}, \dots \in L_u$ and $\{\beta^p, \beta^q, \dots\}, \{\beta'^p, \beta'^q, \dots\}, \dots \in L_u$, we can define the following two kinds of utilitarian capability evaluation rules.

(Definition) *population share-weights utilitarian CER (pU-CER):*²⁵

- (i) For any $C'_i, C_i \in \mathfrak{C}$, for any $\{\alpha^p\}, \{\alpha'^p\} \in L_u$ corresponding to C'_i, C_i , respectively, $C'_i \succsim C_i$, if and only if $\sum_{p \in [0,1]} w^p \widehat{\alpha'^p} \geq \sum_{p \in [0,1]} w^p \widehat{\alpha^p}$, where $w^p = \frac{|N^p|}{|N|}$, $\alpha'^p \in C'_i$, $\alpha^p \in C_i$.
- (ii) For any $C'_i, C_i \in \mathfrak{C}$, for any $\{\beta'^p\}, \{\beta^p\} \in L_u$ corresponding to C'_i, C_i , respectively, $C'_i \succsim C_i$, if and only if $\sum_{p \in [0,1]} w^p \widehat{\beta'^p} \geq \sum_{p \in [0,1]} w^p \widehat{\beta^p}$, where $w^p = \frac{|N^p|}{|N|}$, $\beta'^p \in C'_i$, $\beta^p \in C_i$.

pU-CER aggregates $\widehat{\alpha^p}$ (resp. $\widehat{\beta^p}$) across type p , multiplied by the population share weight w^p . Then, it evaluates the greater is better.

(Definition) *equal-weight utilitarian CER (eU-CER):*

Let us denote the number of types by $|p|$.

²⁵ Here, we assume that the population of each type remains constant between C'_j and C_j . In cases where the population of each type changes, the weight should change to reflect the population of each type.

- i) For any $C_i', C_i \in \mathfrak{C}$, for any $\{\alpha'^p\}, \{\alpha^p\} \in L_u$ corresponding to C_i', C_i , respectively, $C_i' \succsim C_i$, if and only if $\frac{1}{|p|} \sum_{p \in [0,1]} \widehat{\alpha'^p} \geq \frac{1}{|p|} \sum_{p \in [0,1]} \widehat{\alpha^p}$.
- ii) For any $C_i', C_i \in \mathfrak{C}$, for any $\{\beta'^p\}, \{\beta^p\} \in L_u$ corresponding to C_i', C_i , respectively, $C_i' \succsim C_i$, if and only if $\frac{1}{|p|} \sum_{p \in [0,1]} \widehat{\beta'^p} \geq \frac{1}{|p|} \sum_{p \in [0,1]} \widehat{\beta^p}$.

eU-CER aggregates $\widehat{\alpha^p}$ (resp. $\widehat{\beta^p}$) across type p , added with equal weight regardless of the share of population of each type, then evaluates the greater is better.

Next, based on type-dependent informational profiles, which are ordinal *level* comparable, such as $\{\alpha^p, \alpha^q, \dots\}, \{\alpha'^p, \alpha'^q, \dots\}, \dots \in \bar{L}$ and $\{\beta^p, \beta^q, \dots\}, \{\beta'^p, \beta'^q, \dots\}, \dots \in \bar{L}$, we can define maximum values and minimum values in an interval $p \in [0, 1]$ such as $\max_{p \in [0, 1]} \alpha^p$ and $\min_{p \in [0, 1]} \alpha^p$.

These are numerical representation of $p_V^{\max} \in [0, 1]$ and $p_V^{\min} \in [0, 1]$, which are previously defined as $[\#p' \in [0, 1], p'Vp^{\max}]$ and $[\#p' \in [0, 1], p^{\min}Vp']$, respectively (in section 4). Similarly, we can define $\max_{p \in [0, 1]} \beta^p$ and $\min_{p \in [0, 1]} \beta^p$, respectively.

With the assumption of ordinal level comparability and with these concepts, let us define three capability evaluation rules in order.

First, *Rawlsian* capability evaluation rule is defined as follows.

(Definition) Rawlsian CER (R-CER):

- i) For any $C_i', C_i \in \mathfrak{C}$, for any $\{\alpha'^p\}, \{\alpha^p\} \in \bar{L}$, corresponding to C_i', C_i , respectively, $C_i' \succsim C_i$, if and only if $\min_{p \in [0, 1]} \widehat{\alpha'^p} \geq \min_{p \in [0, 1]} \widehat{\alpha^p}$.
- ii) For any $C_i', C_i \in \mathfrak{C}$, for any $\{\beta'^p\}, \{\beta^p\} \in \bar{L}$, corresponding to C_i', C_i , respectively, $C_i' \succsim C_i$, if and only if $\max_{p \in [0, 1]} \widehat{\beta'^p} \leq \max_{p \in [0, 1]} \widehat{\beta^p}$.

R-CER aggregates $\widehat{\alpha^p}$ (resp. $\widehat{\beta^p}$) across type p , giving full weight to the α^p -minimum (resp. β^p - maximum) in all types, which might be different before and after a change, then evaluates the greater is better (resp. the smaller is better).

Second, we define $\{\alpha^p\}$ -maximax -CER, which aggregates $\widehat{\alpha^p}$ across type $p \in [0, 1]$, giving full weight to the maximum in all types, then evaluates the greater is better.

(Definition) $\{\alpha^p\}$ -maximax -CER:

For any $C_i', C_i \in \mathfrak{C}$, for any $\{\alpha'^p\}, \{\alpha^p\} \in \bar{L}$ corresponding to C_i', C_i , respectively, $C_i' \succsim C_i$, if and only if $\max_{p \in [0, 1]} \widehat{\alpha'^p} \geq \max_{p \in [0, 1]} \widehat{\alpha^p}$.

In words, *Rawlsian* capability evaluation rule aims to maximize the α^p -minimum for all $p \in [0, 1]$, while α^p -maximax capability evaluation rule aims to maximize the α^p -maximum value for all $p \in [0, 1]$.

Third, let us define $\{\beta^p\}$ -*minimal*-CER, the capability evaluation rule that maximizes

the number of the set of minimum values in an interval $p \in [0, 1]$, $|M(\min_{p \in [0, 1]} \{\widehat{\beta^p}\})|$.²⁶

(Definition) $\{\beta^p\}$ -minimal -CER:

For any $C'_i, C_i \in \mathfrak{C}$, for any $\{\beta'^p\}, \{\beta^p\} \in \bar{L}$ corresponding to C'_i, C_i respectively, $C'_i \succsim C_i$, if and only if $|M(\min_{p \in [0, 1]} \{\widehat{\beta'^p}\})| \geq |M(\min_{p \in [0, 1]} \{\widehat{\beta^p}\})|$.

In words, $\{\beta^p\}$ -minimal -CER aggregates $\widehat{\beta^p}$ across type $p \in [0, 1]$, giving full weight to the number of the set of minimum values, then evaluates the greater is better.

6.2 Normative Properties of pU-CER, eU-CER and R-CER

Let us introduce several notations and axioms to clarify the similarities and the differences among the above five CERs. First, we present three axioms that are crucial in examining the normative properties of the five rules. For simplicity, descriptions related to β^p are omitted except in cases where α^p and β^p cannot be described as isomorphic.

(Definition) type decomposability axiom:

For any $C'_i, C_i \in \mathfrak{C}_i$, for any $\{\alpha'^p\}, \{\alpha^p\} \in L_u$ corresponding to C'_i, C_i , respectively, if $\alpha^p \geq \alpha^q, \alpha'^p = \alpha^p, \alpha'^q = \alpha^q, \frac{|N'^p|}{|N^j|} > \frac{|N^p|}{|N^j|}$, then $C'_i \succ C_i$.

In words, for any two types $p, q \in [0, 1]$ with different α^p values ($\alpha^p \geq \alpha^q$), if the population ratio of the type with larger value increases (conversely, the ratio of the type with smaller value decreases), holding other values constant, this axiom evaluates capability as improved. In contrast, the following axiom evaluates capability as unchanged in this case.

(Definition) impartiality for types axiom:

For any $C'_i, C_i \in \mathfrak{C}_i$, for any $p, p', q, q' \in [0, 1]$, for any $\{\alpha'^p\}, \{\alpha^p\} \in \bar{L}$ corresponding to C'_i, C_i , respectively, if $\widehat{\alpha'^p} \geq \widehat{\alpha^p}, \widehat{\alpha'^q} \leq \widehat{\alpha^q}, |\widehat{\alpha'^p} - \widehat{\alpha^p}| \geq |\widehat{\alpha'^q} - \widehat{\alpha^q}|$ and, $\widehat{\alpha'^k} = \widehat{\alpha^k}$ for all $k, k' \in [0, 1] \setminus p, p', q, q'$, then $C'_i \succsim C_i$.

From a normative perspective, the interesting point is that the type decomposability axiom permits a partial weighting to each type according to its population ratio, which implies giving an equal weight to any individual regardless the type to which he/she belong. In contrast, impartiality for types axiom give an equal weight to every type regardless of the share of population of each type, while it assigns different weights to individuals. For

²⁶ $|M(\min_{p \in [0, 1]} \{\widehat{\beta^p}\})|$ is a subset of $Min([0, 1], V)$ defined in Section 4.

example, individuals belonging to smaller population groups are assigned greater weights than individuals belonging to larger population groups.

(Definition) Non-dominating Prior (Sen-Hammond equity) axiom:

- i) **α -Non-dominating Prior axiom:** For any $C'_i, C_i \in \mathfrak{C}_i$, for any $\{\alpha'^p\}, \{\alpha^p\} \in \bar{L}$ corresponding to C'_i, C_i , respectively, if $\widehat{\alpha^p} \geq \widehat{\alpha'^p} \geq \widehat{\alpha'^q} \geq \widehat{\alpha^q}$ and for all $k \in [0, 1] \setminus p, q$, $\widehat{\alpha^k} = \widehat{\alpha'^k}$, then $C'_i \succsim C_i$.
- ii) **β -Non-dominating Prior axiom:** For any $C'_i, C_i \in \mathfrak{C}_i$, for any $\{\beta'^p\}, \{\beta^p\} \in \bar{L}$ corresponding to C'_i, C_i , respectively, if $\widehat{\beta^p} \leq \widehat{\beta'^p} \leq \widehat{\beta'^q} \leq \widehat{\beta^q}$ and for all $k \in [0, 1] \setminus p, q$, $\widehat{\beta^k} = \widehat{\beta'^k}$, then $C'_i \succsim C_i$.

(Definition) Non-dominated Prior (Meritocracy) axiom:

- i) **α -Non-dominated Prior axiom:** For any $C'_i, C_i \in \mathfrak{C}_i$ for any $\{\alpha'^p\}, \{\alpha^p\} \in \bar{L}$ corresponding to C'_i, C_i , respectively, if $\widehat{\alpha^p} \geq \widehat{\alpha'^p} \geq \widehat{\alpha'^q} \geq \widehat{\alpha^q}$ and for all $k \in [0, 1] \setminus p, q$, $\widehat{\alpha^k} = \widehat{\alpha'^k}$, then $C_i \succsim C'_i$.
- ii) **β -Non-dominated Prior axiom:** For any $C'_i, C_i \in \mathfrak{C}_i$, for any $\{\beta'^p\}, \{\beta^p\} \in \bar{L}$ corresponding to C'_i, C_i , respectively, for any $\beta'^p, \beta'^q \in C'_i$, $\beta^p, \beta^q \in C_i$, if $\widehat{\beta^p} \leq \widehat{\beta'^p} \leq \widehat{\beta'^q} \leq \widehat{\beta^q}$ and for all $k \in [0, 1] \setminus p, q$, $\widehat{\beta^k} = \widehat{\beta'^k}$, then $C_i \succsim C'_i$.

When conflicts arise between two types, a higher-level type and a lower-level type, Non-dominating Prior (Sen-Hammond equity) axiom requires giving priority to the lower-level type. For example, if the value of a higher-level type decreases but that of a lower-level type increases, the capability is considered to have improved. In contrast, Non-dominated Prior axiom requires giving priority to the higher-level type in the similar conflicting situation.

Next, we note the axioms familiar from the social welfare function, re-formalize according to the framework of this study.

(Definition) anonymity axiom:

If $\rho(p)$ is a permutation function applied to the set of types, and if for any $C'_i, C_i \in \mathfrak{C}_i$, for any $\{\alpha'^p\}, \{\alpha^p\} \in \bar{L}$ corresponding to C'_i, C_i , respectively, for any $\alpha^p \in C_i$ and $\alpha'^{\rho(p)} \in C'_i$, if $\widehat{\alpha^p} = \widehat{\alpha'^{\rho(p)}}$, then $C'_i \sim C_i$.

(Definition) weak Pareto axiom:

For any $C'_i, C_i \in \mathfrak{C}_i$, for any $\{\alpha'^p\}, \{\alpha^p\} \in \bar{L}$ corresponding to C'_i, C_i , respectively, if for any $p \in [0, 1]$, $\widehat{\alpha'^p} > \widehat{\alpha^p}$, then $C'_i \succ C_i$.

(Definition) *strong Pareto axiom:*

For any $C'_i, C_i \in \mathfrak{C}_i$, for any $\{\alpha'^p\}, \{\alpha^p\} \in \bar{L}$ corresponding to C'_i, C_i , respectively, if for any $p \in [0, 1], \widehat{\alpha'^p} \geq \widehat{\alpha^p}$ and for some $q \in [0, 1], \widehat{\alpha'^q} > \widehat{\alpha^q}$, then $C'_i \succ C_i$.

Based on these various axioms, the following propositions summarize the similarities and differences among the above five capability evaluation rules. Proofs of each proposition is in Appendix Proof, respectively.

Proposition 2: pU-CER satisfies anonymity, strong Pareto, the type decomposition axiom, while pU-CER does not satisfy impartiality for types axiom, Non-dominating Prior (Sen-Hammond equity) axiom, Non-dominated Prior axiom, the well-being freedom axiom and the rational freedom axiom.

Proposition 3: eU-CER satisfies anonymity, strong Pareto, impartiality for types axiom but does not satisfy the type decomposition axiom, Non-dominating Prior (Sen-Hammond equity) axiom, Non-dominated Prior axiom, the rational freedom axiom and the well-being freedom axiom.

Proposition 4: R-CER satisfies Anonymity, weak Pareto, Non-dominating Prior (Sen-Hammond equity) axiom, furthermore, R-CER satisfies the well-being freedom axiom. But R-CER does not satisfy strong Pareto, impartiality for types axiom, the type decomposition axiom, Non-dominated Prior axiom and the rational freedom axiom.

Proposition 5: $\{\alpha^p\}$ -maximax-CER satisfies Anonymity, weak Pareto, the well-being freedom axiom and Non-dominated Prior axiom but does not satisfy strong Pareto, impartiality for types axiom and the type decomposition axiom, Non-dominating Prior (Sen-Hammond equity) axiom, and the rational freedom axiom.

Proposition 6: $\{\beta^p\}$ -minimal-CER satisfies Anonymity, weak Pareto, the rational freedom axiom but does not satisfy the well-being freedom axiom, strong Pareto, impartiality for types axiom and the type decomposition axiom, Non-dominating Prior (Sen-Hammond equity) axiom, Non-dominated Prior axiom and the well-being freedom axiom.

The next table 3 summarizes Proposition 2 to 6 that examine normative properties of CERs. To save the space, we add the actual measurement of size and unfreedom of capability

sets for each category based on each aggregation rule (see Section 7 for the discussion).

Table 3: Summary of CERs, Axioms and Actual Measurement of Size/Unfreedom of Capability Sets for Each Category Based on Each Aggregation Rule.

	space	rational freedom	well-being freedom	anonymity	weak Parato	strong Parato	impartiality of types	group- decompositi on	non- dominating	non- dominated	measurement by data			
												category-CR (public care recipients)	category-WD (people with disability)	category-GE (general elderly)
pU-CER	L_U	no	no	YES	YES	YES	no	YES	no	no	total size	0.53	0.57	0.64
											unfreedom	0.06	0.06	0.03
eU-CER	L_U	no	no	YES	YES	YES	YES	no	no	no	total size	0.52	0.53	0.62
											unfreedom	0.14	0.15	0.10
R-CER	$\bar{L}$	no	YES	YES	YES	no	no	no	YES	no	total size	0.13	0.30	0.43
											unfreedom	0.35	0.70	0.60
$\{\alpha^p\}$ -maximax	$\bar{L}$	no	YES	YES	YES	no	no	no	no	YES	total size	1.00	0.93	0.90
$\{\beta^p\}$ -minimal	$\bar{L}$	YES	no	YES	YES	no	no	no	no	no	unfreedom	0.39	0.35	0.33
pooled OLS		—	—	—	—	—	—	—	—	—	total size	0.50	0.53	0.62
											unfreedom	0.05	0.05	0.04

Notes: Proposition 2 to 6. pU-CER means population-share weighted utilitarian aggregation rule, eU-CER means equal weighted utilitarian aggregation rule, and R-CER means Rawlsian aggregation rule. α -max and β -min indicate $\{\alpha^p\}$ -maximax-CER and $\{\beta^p\}$ -minimal-CER, respectively. The measurement is the results of authors' calculations based on Appendix Table 1. $\bar{L}$, and L_u indicate ordinal level inter-personal comparability, and cardinal unit inter-personal comparability, respectively.

6.3 Remarks on freedom and equality (in the context of liberalism)

Finally, we make three remarks on the intrinsic relationship between freedom and equality. First, in our context, wellbeing freedom requires equality. In the ATE method adopted as the type aggregation rule in this paper, the size α^p is defined as a decreasing function of the difference β^p between two actions (going-out and staying home). In this case, $\{\alpha^p\}$ -maximax CER, which requires maximizing the well-being freedom, simultaneously incorporates the equality principle of avoiding the difference (a typical example being maximizing the minimum value between actions). In contrast, in the other type aggregation rules, such as in which α^p is defined as an increasing function of the difference β^p (e.g. α^p has coefficients of 1 for the total and -1 for the difference), the $\{\alpha^p\}$ -maximax CER deviates from the equality principle of avoiding the difference and, the contrast with R-CER, which seeks to maximize the minimum value of α^p , becomes more evident²⁷.

Second, the maximization of the number of the minimal set in the second part of the definition of the rational freedom is usually understood as a criterion avoiding negative values. A typical example is the principle of maximizing the number of the minimal set of (alongside the level of) the difference β^p . From view of this example, we recognize that this definition is plausible in terms of equality as well as freedom. Additionally, We must note that this paper’s definition of rational freedom leaves room for accepting the principle of “maximization of the number of the minimal set” of the *size* α^p , which embodies a positive value to pursue. This might be counterintuitive. Nevertheless, recalling that this principle allows for the interpretation that it seeks a state where no one dominates anyone else, it might be supported from equality as well as freedom.

Third, Rawlsian principle is usually regarded as an equality criterion which is exactly the opposite of the principle of freedom. However, as shown by Proposition 4, R-CER that is based on Rawlsian principle satisfies well-being freedom. This means that one cares his/her own well-being freedom without contracting with equality. Similarly, $\{\beta^p\}$ -minimal-CER is taken as typical principle of equality, because it satisfies intra-type equality (minimizing the difference β^p between two actions) and inter-type equality (maximizing the number of the minimal set of β^p). However, proposition 6 says $\{\beta^p\}$ -minimal-CER satisfies rational freedom. This also means that one can care his/her own rational freedom without contracting equality.

Within our aggregation rules, “freedom does not contradict equality” ; instead, the capability framework uncovers an unexpected affinity between the two. More precisely, pU-CER/eU-CER is silent on inequality, since utilitarian criteria address only aggregation, not distribution. Under these evaluation rules, we describe only the size of capability sets and the degree of unfreedom they embody. This implies merely that, in terms of unfreedom, Category-CR exceeds Category-GE; it does not imply that Category-CR involves greater inequality than Category-GE. Conversely, R-CER/max-max/minimal explicitly captures inequality (within a group). By definition, when the capability size of Category-CR is smaller than that of Category-WD, this is equivalent to saying that, in the Rawlsian sense, Category-CR exhibits greater inequality in capability than Category-WD.

²⁷ Along the ATE context (alpha is the decreasing function of beta), we do not consider of this situation.

7 Comparable Measurement of Capability Sets

We aggregate the type-specific components into category-level capability sets using the CERs defined in Section 6. The results, summarized in Table 3, highlight how different normative judgments alter the assessment of group advantage in terms of capability. For example, regarding the size of the capability set, under the pU-CER, Category-WD appears to have larger capabilities than Category-CR. Conversely, under the eU-CER, this relationship is more ambiguous. Therefore, the ranking between Category-CR and Category-WD is sensitive to the aggregation rule, even among utilitarian rules, whereas Category-GE exhibits the largest size consistently under the pU-CER, the eU-CER, and the R-CER.

These patterns have a clear interpretation. The evaluation order of capability sets is driven by the combination of (1) distributional differences in specific components ($\widehat{\alpha^p}$ and N^p , in our case) across categories and (2) the normative weighting inherent in the evaluation rule. Under the pU-CER, the majority type dominates the results of evaluation. Consequently, when a larger group (even if a minority is severely constrained) achieves more functionings, the entire category is regarded as having a larger capability set. In contrast, the eU-CER treats each type equally, giving “hidden” or rare types full weight. As a result, the eU-CER reveals problems in small subgroups that the pU-CER might overlook, leading to a result in which Category-CR is almost equivalent off than Category-WD.

A specific example of this phenomenon is observed in Category-WD, where the size of the capability set decreases from 0.57 to 0.53 when shifting the rule from the pU-CER to the eU-CER. This outcome reflects the normative trade-off. When we emphasize the formal equality among persons (e.g. one vote for one person), pU-CER concludes category-WD prepares for larger size of capability than category-CR. On the other, when we prioritizes group representation (e.g. one vote for one country), eU-CER concludes category-WD prepares for almost the same size of capability as category-CR.

This implies the distribution of $\widehat{\alpha^p}$ is positively correlated with N^p in Category-WD, whereas these are not so correlated in Category-CR. This difference in distribution of components may result in different judgement; larger $\widehat{\alpha^p}$ type is more weighted in pU-CER, while there is small $\widehat{\alpha^p}$ type, even though its N^p is small, that is more weighted in eU-CER. On the other hand, given the distribution of Category-CR, such a change in evaluation rule does not significantly alter the size of capability sets.

This property is most pronounced when evaluating the use of R-CER, α^p -maximax, and β^p -minimal rules. R-CER which emphasizes the minimum values within a group, yields lower absolute capability levels but preserves the relative ranking observed under other CER rules. In contrast, the α^p -maximax criterion, which considers only the highest achievement in a group, produces a completely opposite evaluation: under this rule, Category-CR appears to have the largest capability set. This contrast reveals a key structural characteristic of the data. The welfare of the most disadvantaged types (the “minimum”) tends to move in tandem with that of the average type, whereas the welfare of the most advantaged types (the “maximum”) does not show this correlation. Consequently, when the normative focus shifts between the average and the minimum, the overall level of assessment changes, but the relative disadvantage of groups remains robust. However, when the focus shifts to the maximum, the evaluation diverges sharply from other rules.

Unlike capability size, the measurement of unfreedom shows greater variation across

CERs, reflecting the complex nature of constraint. While Category-GE consistently shows the lowest unfreedom, the comparison between Category-CR and Category-WD is nuanced. Category-WD exhibits greater unfreedom under the R-CER. Normatively, this implies that the “most unfree” individuals among the disabled population face significantly higher coercion—larger gaps between doing ‘X’ and ‘not X’—than their counterparts in the public care group. While the average unfreedom, based on pU-CER, might be similar between the two groups, the Rawlsian lens exposes the depth of constraint faced by the most vulnerable segment within the disabled category.

The last row of Table 3 shows a pooled regression that treats each group as if it had one homogeneous type. This “average” approach produces results similar to the utilitarian CERs (pU and eU). However, this method smooths over critical heterogeneity. In effect, the pooled model cannot distinguish between a society where everyone is moderately free (each group is similar on average), versus a society where most people are free but a minority is severely constrained (high inequality within the group). Our type-based methodology makes this distinction. By disaggregating into multiple types and then re-aggregating, we preserve information about even the smallest, most disadvantaged subgroups. This process integrates each group’s internal differences into a single capability metric. In summary, the pooled (average-type) view misses hidden inequality, whereas the proposed approach captures it.

8 Concluding Remark

This study addresses a long-standing empirical challenge in the Capability Approach (CA): the measurement of capability sets, which requires evaluating the welfare of counterfactual, unchosen alternatives. By interpreting the average treatment effect (ATE) from a fixed-effects panel model as a consistent estimator of counterfactual welfare levels, we operationalize an otherwise abstract theoretical construct and provide a unified framework that connects normative social choice theory with modern causal inference.

The axiomatic analysis clarifies that the type-specific ATE captures two distinct dimensions of freedom. The first, well-being freedom, corresponds to the maximum achievable functioning level and thus to the size of the capability set. The second, rational freedom, reflects the welfare asymmetry across available options and provides a measure of unfreedom or coercion—how strongly individuals are compelled toward a particular action to avoid substantial welfare loss. This distinction allows welfare evaluation to move beyond the mere feasibility of options and to account for the structural constraints embedded in the choice environment.

Our empirical application to elderly individuals in Japan illustrates a central insight of the CA. High observed frequencies of a particular activity—such as frequently going out—do not necessarily indicate greater freedom. Instead, such behavior may arise from extreme welfare asymmetry, effectively collapsing the capability set to a single viable option. Evaluating well-being therefore requires assessing the full opportunity set rather than relying solely on achieved functionings.

Comparisons across groups under alternative normative aggregation rules further demonstrate that welfare rankings are highly sensitive to the interaction between distributional het-

erogeneity in underlying types and the normative weights imposed by each rule. Population-weighted utilitarian evaluation favors majority types, whereas equal-weighted utilitarian and Rawlsian criteria reveal severe, previously hidden deprivations among minority lifestyle types. In particular, the Rawlsian evaluation indicates that the most constrained individuals within the “With Disabled” category experience greater unfreedom than the most disadvantaged individuals in the “Public Care Receiver” category, while simultaneously highlighting that the minimum capability set size is substantially smaller in the latter group. These findings underscore the importance of accounting for intra-group heterogeneity in distribution-sensitive welfare analysis and suggest that policy priorities — such as subsidies aimed at encouraging out-going behavior — depend critically on the normative principles adopted.

Several scope conditions of the present analysis also point to directions for future research. First, we take the set of ten functionings as given, following established precedents in the literature and considerations of empirical feasibility. As is well known, however, the selection of relevant functionings is inherently normative and remains a central point of contention in multidimensional welfare analysis. Our results should therefore be interpreted as conditional on this specification. Relatedly, we summarize multidimensional outcomes using a simple average, which ensures transparency but abstracts from differential weights and interactions across dimensions. Examining alternative lists and aggregation rules would allow a richer representation of well-being.

Second, our approach relies on the identification of the ATE as a valid approximation of counterfactual welfare levels. When this assumption is violated — for example, due to omitted variables or selection on unobservables — welfare comparisons may be distorted. Extending the framework to settings with partial identification, sensitivity analysis, or alternative causal estimands is a natural extension. Moreover, while we examine the normative properties of estimators that satisfy a set of normative axioms, we do not establish uniqueness; identifying the full class of admissible estimators and their empirical trade-offs remains an open question.

Finally, the analysis is restricted to binary treatment environments, and generalizing the axiomatic framework to multi-valued or continuous treatments represents an important direction for future work. Overall, by grounding the empirical measurement of opportunity sets in both causal inference and normative theory, this study advances the Capability Approach from a largely philosophical framework toward a rigorous and empirically implementable tool for welfare analysis.

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Appendix A: Summary of Type-by-Type Regression Results

PANEL A: Group-CR																					
VARIABLES	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)	(15)	(16)	(17)	(18)	(19)	(20)	
	function_all	function_all	function_all	function_all	function_all	function_all	function_all	function_all	function_all	function_all	function_all	function_all	function_all	function_all	function_all	function_all	function_all	function_all	function_all	function_all	
out		0.250 (0.000)	-0.142 (0.180)	-0.060 (0.102)	-0.009 (0.020)	-0.150 (0.000)	-0.007 (0.072)	0.250 (0.156)	-0.350 (0.000)	0.184 (0.090)	0.320 (0.143)	-0.008 (0.057)	-0.000 (0.121)	0.263 (0.124)	0.175 (0.135)	0.320 (0.598)	-0.100 (0.000)	-0.089 (0.394)	-0.125 (0.179)		
Constant	0.408 (0.029)	0.450 (0.000)	0.506 (0.106)	0.378 (0.095)	0.925 (0.017)	1.000 (0.000)	0.536 (0.062)	0.525 (0.132)	0.700 (0.000)	0.463 (0.094)	0.385 (0.097)	0.591 (0.085)	0.555 (0.149)	0.287 (0.175)	0.395 (0.163)	0.130 (0.658)	0.400 (0.000)	0.564 (0.436)	0.700 (0.219)	0.571 (0.037)	
Observations	118	11	20	27	10	11	42	18	12	45	14	32	15	22	24	14	10	22	24	228	
R-squared	0.099	1.000	0.316	0.255	0.997	1.000	0.119	0.738	1.000	0.318	0.853	0.322	0.826	0.649	0.470	0.449	1.000	0.410	0.377	0.070	
Number of id	51	1	5	9	2	1	13	2	1	11	2	4	2	2	3	2	1	2	2	66	
type	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	
ratio	0.00	0.18	0.25	0.33	0.40	0.45	0.50	0.56	0.58	0.67	0.71	0.75	0.80	0.82	0.83	0.86	0.90	0.91	0.92	1.00	

PANEL B: Group-WD																						
VARIABLES	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)	(15)	(16)	(17)	(18)	(19)	(20)	(21)	(22)
	function_all	function_all	function_all	function_all	function_all	function_all	function_all	function_all	function_all	function_all	function_all	function_all	function_all	function_all	function_all	function_all	function_all	function_all	function_all	function_all	function_all	function_all
out		0.400 (0.000)	-0.023 (0.100)	-0.114 (0.119)	-0.200 (0.259)	0.026 (0.060)	0.150 (0.000)	0.150 (0.000)	0.062 (0.074)	0.195 (0.078)	0.142 (0.145)	0.100 (0.000)	0.066 (0.052)	0.057 (0.117)	-0.113 (0.115)	0.088 (0.172)	0.063 (0.084)	0.213 (0.186)	0.400 (0.000)	-0.700 (0.302)	0.139 (0.075)	
Constant	0.423 (0.033)	0.350 (0.000)	0.356 (0.083)	0.669 (0.102)	0.300 (0.224)	0.444 (0.059)	0.750 (0.000)	0.400 (0.000)	0.322 (0.122)	0.433 (0.073)	0.625 (0.177)	0.700 (0.000)	0.329 (0.074)	0.768 (0.225)	0.654 (0.125)	0.617 (0.115)	0.565 (0.107)	0.508 (0.173)	0.300 (0.000)	0.925 (0.213)	0.679 (0.088)	0.637 (0.025)
Observations	83	11	20	28	14	57	10	12	36	44	24	11	78	22	64	33	60	24	10	22	60	498
R-squared	0.374	1.000	0.351	0.510	0.534	0.335	1.000	1.000	0.297	0.431	0.385	1.000	0.313	0.523	0.147	0.581	0.215	0.285	1.000	0.630	0.193	0.018
Number of id	37	1	4	4	2	11	1	1	4	16	2	1	14	2	9	3	6	3	1	2	5	110
type	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
ratio	0.00	0.18	0.20	0.25	0.29	0.33	0.40	0.42	0.44	0.50	0.58	0.64	0.67	0.73	0.75	0.82	0.83	0.88	0.90	0.91	0.92	1.00

PANEL C: Group-GE																						
VARIABLES	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)	(15)	(16)	(17)	(18)	(19)	(20)	(21)	(22)
	function_all	function_all	function_all	function_all	function_all	function_all	function_all	function_all	function_all	function_all	function_all	function_all	function_all	function_all	function_all	function_all	function_all	function_all	function_all	function_all	function_all	function_all
out		0.000 (0.000)	0.325 (0.068)	0.305 (0.133)	-0.602 (0.104)	0.087 (0.067)	-0.163 (0.146)	-0.113 (0.189)	0.125 (0.089)	0.044 (0.042)	-0.200 (0.000)	0.100 (0.173)	0.029 (0.063)	-0.033 (0.027)	-0.015 (0.263)	-0.155 (0.079)	-0.103 (0.083)	0.026 (0.031)	0.083 (0.100)	0.056 (0.128)	-0.043 (0.138)	0.085 (0.031)
Constant	0.479 (0.025)	0.750 (0.000)	0.525 (0.048)	0.483 (0.103)	0.635 (0.103)	0.450 (0.075)	0.712 (0.117)	0.629 (0.189)	0.568 (0.095)	0.900 (0.044)	0.750 (0.150)	0.433 (0.083)	0.755 (0.046)	0.433 (0.327)	0.753 (0.126)	0.554 (0.114)	0.484 (0.114)	0.582 (0.036)	0.442 (0.158)	0.763 (0.191)	0.596 (0.178)	0.485 (0.052)
Observations	110	12	22	21	25	36	28	22	21	138	11	16	45	108	14	27	33	208	22	27	21	144
R-squared	0.355	1.000	0.844	0.528	0.789	0.398	0.348	0.438	0.629	0.097	1.000	0.699	0.218	0.112	0.434	0.565	0.496	0.058	0.613	0.255	0.483	0.240
Number of id	61	1	2	3	4	4	5	2	3	30	1	2	7	9	2	3	3	53	2	3	3	12
type	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
ratio	0.00	0.08	0.09	0.14	0.20	0.22	0.25	0.27	0.29	0.33	0.36	0.38	0.40	0.42	0.43	0.44	0.45	0.50	0.55	0.56	0.57	0.58

(23)	(24)	(25)	(26)	(27)	(28)	(29)	(30)	(31)	(32)	(33)	(34)	(35)	(36)	(37)	(38)	(39)	(40)	(41)
function_all	function_all	function_all	function_all	function_all	function_all	function_all	function_all	function_all	function_all	function_all	function_all	function_all	function_all	function_all	function_all	function_all	function_all	function_all
0.070 (0.055)	-0.200 (0.127)	0.049 (0.032)	0.016 (0.028)	-0.075 (0.196)	0.124 (0.046)	0.096 (0.047)	0.016 (0.027)	0.114 (0.047)	0.014 (0.045)	0.163 (0.038)	0.045 (0.038)	0.124 (0.038)	0.045 (0.093)	-0.022 (0.076)	0.044 (0.080)	0.027 (0.117)	0.118 (0.056)	0.115 (0.037)
0.676 (0.081)	0.775 (0.110)	0.733 (0.053)	0.570 (0.037)	0.550 (0.241)	0.608 (0.058)	0.551 (0.076)	0.586 (0.038)	0.607 (0.063)	0.639 (0.053)	0.435 (0.051)	0.572 (0.037)	0.606 (0.088)	0.629 (0.086)	0.684 (0.095)	0.888 (0.145)	0.888 (0.071)	0.612 (0.046)	0.624 (0.012)
85	16	88	285	20	49	121	276	54	105	154	360	63	64	63	70	187	420	2,711
0.179	0.771	0.152	0.117	0.519	0.455	0.112	0.076	0.339	0.090	0.253	0.068	0.199	0.276	0.129	0.177	0.080	0.054	0.016
11	2	8	47	2	7	11	35	6	14	14	35	9	8	7	7	17	35	468
23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41
0.60	0.63	0.64	0.67	0.70	0.71	0.73	0.75	0.78	0.80	0.82	0.83	0.86	0.88	0.89	0.90	0.91	0.92	1.00

Notes: Authors' calculation.

Appendix B: Proof for Lemma 1

(1) Given the definitions of α^p , which is an increasing function of $F^p(\geq 0)$ and a decreasing function of $\beta^p(\geq 0)$, it is obvious that α^p -maximization rule satisfies the strong Pareto for plural criteria,

(2) The former part is proven as follows. Suppose that $\beta'^p \leq \beta^p$, $F'^p \geq F^p$, and $F'^p > F^p$ but $\beta'^p = \beta^p$. According to strong Pareto axiom, $C'_i \succ C_i$ hold. Yet, according to β^p -minimization rule, $C'_i \succ C_i$ does not necessarily hold. This is a contradiction. The latter part is proven as follows. Suppose that $\beta'^p < \beta^p$ and $F'^p > F^p$, then, according to β^p -minimization rule, $C'_i \succ C_i$ holds. It shows that β^p -minimization rule satisfies the weak Pareto for plural criteria. $\square$

Appendix C: Proof for Lemma 2

It is obvious that, since α^p is an increasing function of F^p and a decreasing function of difference β^p , and an increasing function of the combination of them with a certain weight, we can confirm that α^p -maximization principle satisfies axioms of the well-being Freedom.

If β^p increase, then the number of maximal set $|Max(A, Q^p)|$ decrease, therefor the rational freedom declares that an individual's capability is reduced. Yet, α^p -maximization rule judges that an individual's capability is improved if the increase of the total exceeds the decrease in unfreedom. This implies, it does not satisfy the rational freedom. $\square$

Appendix D: Proof for Lemma 3

(1) Recalling the concept of the maximal set, (1) is obvious. First, if $\beta'^p = \beta^p = 0$, $|Max(A, Q'^p)| = |Max(A, Q^p)| = 2$. If $\beta'^p = 0 < \beta^p$, $|Max(A, Q'^p)| = 2 > |Max(A, Q^p)| = 1$. By combining these two equations, we get a statement to be proven, that is, if $\beta'^p \leq \beta^p$, then $|Max(A, Q'^p)| \geq |Max(A, Q^p)|$.

(2) The former statement of (2) is induced from the definition of the rational freedom axioms, that is, C'_i is at least as free as C_i , if and only if for each $p \in [0, 1]$, $|Max(A, Q'^p)| \geq |Max(A, Q^p)|$ and for all $q \neq p \in [0, 1]$, $|Max(A, Q'^p)| = |Max(A, Q^p)|$.

The later statement of (2) is proved as follows. β^p -minimization principle requires that if $\beta'^p \leq \beta^p$, even if $\zeta F'^p - \eta \beta'^p \leq \zeta F^p - \eta \beta^p$, $C'_i \succsim C_i$ holds, while well-being freedom requires that if $\zeta F'^p - \eta \beta'^p \leq \zeta F^p - \eta \beta^p$, even if $\beta'^p \leq \beta^p$, $C'_i \precsim C_i$ holds. It is a contradiction. $\square$

Appendix E: Proof for Proposition 1

It is obvious that this hold, since $\widehat{\alpha^p} = F^p - \widehat{\beta^p} \cdot \widetilde{P}$ is a variation of $\zeta F'^p - \eta \beta'^p$ with $\zeta = 1, \eta = \widetilde{P}$. The details can be verified as follows.

First, since every individual belonging to the same type has the same going-out ratio, $\widehat{\beta^p}$ in Section 4 can be rewritten as follows,

$$\widehat{\beta^p} = \frac{\sum_{i,t} [(F_{it} - F_i)(OUT_{it} - P_i)]}{\sum_{i,t} [(OUT_{it} - P_i)^2]}.$$

This formula is further simplified as the following way. Let us denote the numerator as NUM , and the denominator as DEN . And let us denote $F_i^O = \frac{1}{T_i^O} \sum_{(t)^O} F_{it}$ and $F_i^S = \frac{1}{T-T_i^O} \sum_{(t)^S} F_{it}$. Then, F_i , in the first term in the numerator (NUM), can be rewritten as follows.

$$\begin{aligned}
F_i &= F_i^O + F_i^S \\
&= \frac{T_i^O}{T} \cdot \frac{1}{T_i^O} \sum_{(t)O} F_{it} + \frac{T - T_i^O}{T} \cdot \frac{1}{T - T_i^O} \sum_{(t)S} F_{it} \\
&= P_i \cdot \frac{1}{T_i^O} \sum_{(t)O} F_{it} + (1 - P_i) \cdot \frac{1}{T - T_i^O} \sum_{(t)S} F_{it} \\
&= P_i F_i^O + (1 - P_i) F_i^S.
\end{aligned}$$

Then, the numerator (NUM) can be rewritten as follows.

$$\begin{aligned}
NUM &= \sum_{i,t} \left[(OUT_{it} - P_i) (F_{it} - F_i) \right] \\
&= \sum_{i,t} \left[(OUT_{it} - P_i) (F_{it} - (P_i F_i^O + (1 - P_i) F_i^S)) \right] \\
&= \sum_{i,t} OUT_{it} F_{it} - \sum_{i,t} P_i F_i^O - \sum_{i,t} P_i F_{it} + \sum_{i,t} P_i^2 F_i^O + \sum_{i,t} P_i (1 - P_i) F_i^S.
\end{aligned}$$

NUM is calculated as the sum of the following two cases.

If $OUT_{it} = 1$, then

$$\begin{aligned}
NUM &= \sum_{i,t} F_{it}^O - P_i \sum_{i,t} F_i^O - \sum_{i,t} P_i F_{it}^O + \sum_{i,t} P_i^2 F_i^O \\
&= (1 - P_i) \sum_{i,t} F_{it}^O - P_i (1 - P_i) \sum_{i,t} F_i^O,
\end{aligned}$$

where F_{it}^O refers to individual i 's achieved functionings at time t .

If $OUT_{it} = 0$, then

$$NUM = -P_i \sum_{i,t} F_{it}^S + P_i (1 - P_i) \sum_{i,t} F_i^S,$$

where F_{it}^S refers to individual i 's achieved functionings at time t .

Since F_i^O (resp. F_i^S) is the average of individual i 's achieved functionings when going out

(resp. staying home), the following holds:

$$\begin{aligned}\sum_{i,t} F_{it}^O &= \sum_{i,t} F_i^O = N_p T P_i F_i^O, \\ \sum_{i,t} F_{it}^S &= \sum_{i,t} F_i^S = N_p T (1 - P_i) F_i^S.\end{aligned}$$

Then, we can rewrite NUM as follows.

If $OUT_{it} = 1$, then

$$\begin{aligned}NUM &= (1 - P_i) N_p T P_i F_i^O - P_i (1 - P_i) N_p T P_i F_i^O \\ &= (1 - P_i) (1 - P_i) N_p T P_i F_i^O.\end{aligned}$$

If $OUT_{it} = 0$, then

$$\begin{aligned}NUM &= -P_i N_p T (1 - P_i) F_i^S + P_i (1 - P_i) N_p T (1 - P_i) F_i^S \\ &= -P_i N_p T (1 - P_i) (1 - (1 - P_i)) F_i^S \\ &= -N_p T P_i^2 (1 - P_i) F_i^S.\end{aligned}$$

Combining these two cases, NUM is given by

$$\begin{aligned}NUM &= (1 - P_i) (1 - P_i) N_p T P_i F_i^O - N_p T P_i^2 (1 - P_i) F_i^S \\ &= N_p T P_i (1 - P_i) ((1 - P_i) F_i^O - P_i F_i^S).\end{aligned}$$

Similarly, the denominator (DEN) can be rewritten as follows.

$$DEN = \sum_{i,t} \left[(OUT_{it} - P_i)^2 \right].$$

If $OUT_{it} = 1$, then

$$DEN = \sum_{i,t} \left[(1 - P_i)^2 \right].$$

If $OUT_{it} = 0$, then

$$DEN = \sum_{i,t} \left[(0 - P_i)^2 \right].$$

Then,

$$\begin{aligned}
DEN &= \sum_{i,t} (1 - P_i)^2 + \sum_{i,t} P_i^2 \\
&= N_p T P_i (1 - P_i)^2 + N_p T (1 - P_i) P_i^2 \\
&= N_p T P_i (1 - P_i).
\end{aligned}$$

Combining the numerator and denominator, $\widehat{\beta}^p$ is given by the following.

$$\begin{aligned}
\widehat{\beta}^p &= \frac{N_p T P_i (1 - P_i) ((1 - P_i) F_i^O - P_i F_i^S)}{N_p T P_i (1 - P_i)} \\
&= (1 - P_i) F_i^O - P_i F_i^S
\end{aligned}$$

Substituting $\widehat{\beta}^p$, $\widehat{\alpha}^p$ is expressed as follows.

$$\begin{aligned}
\widehat{\alpha}^p &= \widetilde{F} - \widehat{\beta}^p \cdot \widetilde{P} \\
&= P_i F_i^O + (1 - P_i) F_i^S - ((1 - P_i) F_i^O - P_i F_i^S) \widetilde{P} \\
&= P_i (F_i^O + F_i^S \widetilde{P}) - (1 - P_i) (F_i^O \widetilde{P} - F_i^S) \square
\end{aligned}$$

Appendix F: Proof for Proposition 2

It follows self-evidently from the proofs in prior research on distributive justice that pU-CER satisfies anonymity, strong Pareto. That pU-CER satisfies the type decomposition axiom is proved by taking the case of two types as follows.

Let us denote $w^p = \frac{|N^p|}{|N^j|}$, $w'^p = \frac{|N'^p|}{|N^j|}$, $w^q = \frac{|N^q|}{|N^j|}$, $w'^q = \frac{|N'^q|}{|N^j|}$. Then, if $\alpha^p > \alpha^q$, $\alpha'^p = \alpha^p$, $\alpha'^p = \alpha^p$, $|N'^p| \geq |N^p|$ and $|N^p| + |N^q| = |N'^p| + |N'^q|$, $\widehat{\alpha}^p (\frac{|N'^p|}{|N^j|} - \frac{|N^p|}{|N^j|}) + \widehat{\alpha}^q (\frac{|N'^q|}{|N^j|} - \frac{|N^q|}{|N^j|}) > 0$ holds.

This is rewritten such that $\widehat{\alpha}^p (w'^p - w^p) + \widehat{\alpha}^q (w'^q - w^q) > 0$. Then we have $w'^p \widehat{\alpha}^p + w'^q \widehat{\alpha}^q \geq w^p \widehat{\alpha}^p + w^q \widehat{\alpha}^q$. Then, by definition, $C'_i \succ C_i$ holds, which is to be proven. $\square$

Appendix G: Proof for Proposition 3

It follows self-evidently from the proofs in prior research on distributive justice that eU-CER satisfies anonymity, strong Pareto. That eU-CER does not satisfy the type decomposition axiom is proved as follows. by taking the case of two types as follows as follows.

Note that, first, $\widehat{\alpha'}^p + \widehat{\alpha'}^q = |N'^p| \times \widehat{\alpha'}^p \times \frac{1}{|N'^p|} \times \frac{1}{2} + |N^q| \times \widehat{\alpha'}^q \times \frac{1}{|N'^q|} \times \frac{1}{2}$ and $\widehat{\alpha}^p + \widehat{\alpha}^q = |N^p| \times \widehat{\alpha}^p \times \frac{1}{|N^p|} \times \frac{1}{2} + |N^q| \times \widehat{\alpha}^q \times \frac{1}{|N^q|} \times \frac{1}{2}$.

Then we have that $\widehat{\alpha'}^p + \widehat{\alpha'}^q = \widehat{\alpha}^p + \widehat{\alpha}^q$, if $\alpha^p \geq \alpha^q, \alpha'^p = \alpha^p, \alpha'^q = \alpha^q, |N'^p| \geq |N^p|$ and $|N^p| + |N^q| = |N'^p| + |N'^q|$.

Therefore, we have $C'_i \sim C_i$, which implies $C'_i \succ C_i$ does not holds. $\square$

Appendix H: Proof for Proposition 4

It follows self-evidently from the proofs in prior research on distributive justice that R-CER satisfies Anonymity, weak Pareto, Sen-Hammond equity axiom.

That R-CER satisfies the well-being freedom axiom is proved as follows. Given that R-CER regarding α^p requires maximizing the level of the minimum α^p , it is self-evident that it satisfies the well-being freedom. Furthermore, noting that β^p -max represents the negative value being the most, minimizing β^p max implies minimizing the negative value. Therefore, R-CER regarding β^p also satisfies well-being freedom. $\square$

Appendix I: Proof for Proposition 5

$\{\alpha^p\}$ -maximax -CER is the expansion of the α^p -maximization rule defined in Section 4, which requires to maximize α^p for each p, and which satisfy the Well-being freedom. This rule requires to further maximize α^p -maximum among all p, then obviously it satisfies the Well-being freedom. The latter part can be inferred by recalling that $\{\alpha^p\}$ -maximax -CER contrasts with the Rawlsian rule, which requires to minimize the minimum α^p among all p, which satisfies the β - Non-dominating Prior axiom.

Appendix J: Proof for Proposition 6

The former part is obvious. The latter part is almost obvious, since the objective to maximize is not the level of the minimum values, but rather the number of the set of minimum values.