IZA Institute of Labor Economics

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DISCUSSION PAPER SERIES

IZA DP No. 17619

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ISSN: 2365-9793

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ABSTRACT

Demand Uncertainty and the Optimal Number of Export Destinations

We study how demand uncertainty affects risk-neutral firms' number of export destinations when uncertainty is resolved after firms choose their export destinations and output. We show that firms' ability to allocate their output across destinations in response to destination-specific shock realizations provides even risk-neutral firms an incentive to export. Without appealing to firm-country heterogeneity or increasing marginal cost, our framework can explain why firms export to some but not all ex-ante indistinguishable destinations. We also show how, for a given firm productivity, the optimal number of export destinations depends on the correlation of shocks across the home and foreign countries.

JEL Classification: F12, F61

Keywords: international trade, demand uncertainty, risk-neutral firms, optimal number of export destinations

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1 Introduction

Uncertainty is salient in international trade, and at least some of the uncertainty may be resolved only after firms have chosen their export destinations and output. This is evidenced by firms' willingness to pay a premium to delay production until some of the uncertainty is resolved, either by producing in a more costly location that is geographically closer to the point of sale or by paying extra to ship goods faster.[1] At the same time, firms may have some flexibility in allocating their output across destinations in response to shock realizations that occur after production has taken place. For instance, firms hit by a negative domestic shock may increase their sales abroad by exporting some of the goods intended for sale at home.

This timing of events – namely that firms choose their export destinations and output while faced with uncertainty but can mitigate some of the negative impact of this uncertainty by shuffling their output between markets after uncertainty has been resolved – is supported by the negative correlation between home and foreign sales.[2] The negative correlation seems to contradict the fact that firms have a tendency to export to countries that are similar to their own, which should lead to a positive correlation between home and foreign sales.[3] However, this apparent contradiction can be reconciled if sales at home and abroad are not set independently, but in such a way that increased sales in one market can only be achieved by reduced sales in another. That is, when shocks are realized, a firm can no longer change its export destinations or adjust its output. Instead, the firm responds to favorable conditions in one market relative to another by diverting output to the favorable market from the other market.[4]

[1] See Evans and Harrigan (2005) and Hummels and Schaur (2010).

[2] See Almunia, Antràs, Morales and Lopez-Rodriguez (2021) and also Vannoorenberghe (2012), Blum, Claro and Horstmann (2013), Haddad, Lim, Pancaro and Saborowski (2013), Soderbery (2014) and Ahn and McQuoid (2017). However, Berman, Berthou and Héricourt (2015) find a positive correlation.

[3] See Eaton, Kortum and Kramarz (2011).

[4] Some of the papers cited in footnote 2 offer a related interpretation for the negative correlation between

This paper explores how incorporating uncertainty and these timing features into an otherwise standard international-trade framework affects a firm's optimal number of export destinations. We construct a model in which risk-neutral firms with heterogeneous productivity choose their export destinations, if any, and how much output to produce before the realization of destination-specific shocks, but are free to allocate that output across destinations after their realization. As in the Melitz (2003) model, countries are symmetric, there is a per-destination fixed export cost, and firms produce at constant marginal cost. However, we allow the demand for a firm's good in each destination to be perturbed by a firm-destination-specific shock.[5] Since a firm does not know (the full extent of) the shock realizations when it chooses its export destinations, it is the distribution of shocks rather than any particular shock realization that determines the firm's optimal number of export destinations.

In this setting, becoming an exporter confers on a firm flexibility in determining the ultimate destination of the output it chose to produce. Indeed, an exporting firm has the option of selling more at home (and less abroad) when the realized home shock is relatively favorable, and more abroad (and less at home) when the realized home shock is relatively unfavorable. In contrast, a non-exporting firm is forced to sell all its output at home regardless of the shock realization. Adding more export destinations further enhances a firm's flexibility in allocating its output in response to shock realizations. The upshot is that even risk-neutral firms may choose to export or add export destinations not only to access a larger customer base, but also to spread their sources of demand across distinct

home and foreign sales, namely, that production occurs after the realization of shocks but firms face increasing marginal cost. Our result that uncertainty can incentivize firms to export would remain intact in this alternative scenario because an additional market would still increase firms' flexibility in allocating productive capacity across markets in response to shock realizations.

[5] The importance of firm-destination-specific shocks is shown by Eaton, Kortum and Kramarz (2011) and Munch and Nguyen (2014), who find that such shocks (rather than productivity shocks or other firm-specific factors) explain most of the sales variation within particular export markets.

markets.[6] We refer to this as the *ex-post allocation motive for exporting*, since it operates by allowing the firm to maximize the benefit of favorable shock realizations and minimize the loss from unfavorable shock realizations by allocating its chosen output ex post across destinations in response to shock realizations.[7]

We show that the ex-post allocation motive for exporting entails that the counterfactual prediction in the Melitz model of a bang-bang solution (i.e., if countries are symmetric, the per-destination export cost is fixed, and the marginal cost is constant, then firms either do not export or export to all destinations) need not hold. That is, *uncertainty by itself can explain why firms export to some destinations but not others,* even though the destinations are indistinguishable ex ante. The reason is that the marginal benefit from adding more export destinations may, depending on the shock distributions, decrease with the number of destinations. As a consequence, even if a firm finds it worthwhile to export to some destinations, there will, in general, be some number of export destinations beyond which the gain from adding another destination exceeds the fixed cost of exporting. This is the firm's optimal number of export destinations, which increases in the firm's productivity.[8]

To establish that the export solution may not be bang-bang and also illustrate the

[6] Esposito (2022) considers how risk-averse firms may benefit from exporting if demand shocks are imperfectly correlated across destinations. In his analysis the imperfect correlation of shocks has no effect on a firm's expected profit but reduces the variability of the profit. The benefit therefore relies on the risk aversion of the firm. In contrast, in our analysis the imperfect correlation of shocks increases the firm's expected profit, and therefore even risk-neutral firms benefit. Danziger and Danziger (forthcoming) show that if a firm's export cost is uncertain and workers can be laid off, and hence output adjusted after the uncertainty has been resolved, the introduction of a minimum wage may lead a risk-neutral firm to increase its export market. The reason is that the workers' expected income with the minimum wage may exceed their reservation wage. The firm can therefore increase its export market without having to increase the wage.

[7] The ex-post allocation motive for exporting is consistent with Vannoorenberghe (2012) who finds that exporters with small (large) export shares have less (more) volatile total sales than non-exporters. It is also consistent with Fillat and Garetto (2015) and Fillat, Garetto and Oldenski (2015), who find that exporting firms have higher average returns than non-exporting firms.

[8] Papers using firm-level heterogeneity or asymmetry due to, for example, firm-country-specific market demand, trade costs, or cost shifters to explain firms' export decisions include Chaney (2008), Helpman, Melitz and Rubinstein (2008), Arkolakis (2010), Amiti and Davis (2011) and Eaton, Kortum and Kramarz (2011).

importance of shock correlation for the optimal number of export destinations, we assume that the economy has three sectors whose shock distributions differ by how the shocks are correlated across the home and foreign countries:

In one sector the shocks are global, that is, the shock realizations are identical in all foreign countries and proportional to the shock realization in the home country. As all the shocks are perfectly correlated, there is no benefit from being able to shuffle output between destinations after the shock realizations. The ex-post allocation motive for exporting is therefore absent, and a firm either does not export or exports to all of the identical foreign countries.

In another sector, the foreign shocks are common, that is, the shock realizations are identical in all foreign countries and independent of the shock realization in the home country. As the foreign shocks (which are perfectly correlated) are uncorrelated with the home shock, a firm can benefit from shuffling output between the home and foreign markets (but not between the foreign markets) after the shock realizations. We show that the ex-post allocation motive for exporting then ensures that some firms export to some but not all foreign countries. Indeed, there are firms that export to any given number of the identical foreign countries.

In the last sector, the foreign shocks are idiosyncratic, that is, the shock realizations in the foreign countries are independent of each other and of the shock realization in the home country. As all the shocks at home and abroad are uncorrelated, a firm can benefit from shuffling output between the home and foreign countries as well as between the foreign countries after the shock realizations. The ex-post allocation motive for exporting in the sector with idiosyncratic foreign shocks is thus stronger than in the sector with common foreign shocks and may cause firms to export to some but not all of the ex-ante identical foreign countries. However, it is possible that the ex-post allocation motive for exporting is so powerful that firms that export will export to all of the foreign countries.

Having established that the export solution is not necessarily bang-bang, we examine how the increased shock correlation (and hence the weakening of the ex-post allocation motive for exporting) as one moves from the sector with idiosyncratic foreign shocks to the sector with common foreign shocks and then to the sector with global shocks affects a firm's optimal number of export destinations. We find that, for a given firm productivity, in the sector with idiosyncratic foreign shocks, firms will have at least as many export destinations as firms in the sector with common foreign shocks, which themselves will have at least as many export destinations as firms in the sector with global shocks.

Some papers have used uncertainty and volatility to explain the pattern of firms' entry into and exit from export markets.[9] One approach posits that firms face constant but unknown demand, which they can discover only through experimental sales.[10] Since firms may exit the export market if demand turns out to be low, this can explain why some firms temporarily enter export markets before permanently exiting. Another approach posits that demand is volatile and firms only export when foreign demand is high.[11] This can explain why some firms repeatedly enter and exit the same export market.

Our explanation of the impact of uncertainty on firms' export behavior differs from these approaches in three important ways. First, in these papers, home and foreign sales can be adjusted independently. Consequently, the ex-post allocation motive for exporting is absent. Second, in our paper, because the decision to export is made before the shocks are realized, it is the distribution, rather than the realization, of shocks that matters.[12] Third, because we emphasize the role of shock distributions rather than particular shock realizations, our paper focuses on explaining the impact of uncertainty on a firm's long-

[9] Eaton, Kortum and Kramarz (2004), Bernard, Jensen, Redding and Schott (2007) and Eaton, Eslava, Kugler and Tybout (2008) highlight the importance of the churning of firms into and out of export markets.

[10] See Albornoz, Pardo, Corcos and Ornelas (2012) and Nguyen (2012).

[11] See Vannoorenberghe, Wang and Yu (2016). Blum, Claro and Horstmann (2013) show that in the presence of increasing costs a negative shock realization at home can incentivize firms to export.

[12] In the case of experimental exporting, the initial decision to experiment is made under uncertainty, but the subsequent decision of whether to continue exporting is made after uncertainty is resolved.

term export status rather than the impact of uncertainty on the constant churning of firms into and out of export markets.

The few papers that consider how the distribution of shocks influences a firm's decision to export and its choice of export destinations assume that production can be fully adjusted after the realization of shocks.[13] So only the "investment" decision of whether or where to export is made under uncertainty, while the actual production decision is made optimally after uncertainty has been resolved. With constant marginal cost, this implies that each foreign country can be considered in isolation so that a decision to export to one particular foreign country is independent of home-market circumstances and the decision to export to any other foreign country. Therefore, if foreign countries are identical and face the same shock distribution, a firm will either not export at all or export to all foreign countries. Similarly to the Melitz model, no firm will export to only some foreign countries.

In contrast to these papers, our focus is on an environment in which shocks can alter demand between the time that production takes place and the time that goods are shipped or sold. This timing is arguably relevant in almost every industry since firms always face a positive probability that demand will change abruptly, and hence there is always some demand uncertainty remaining even after production has taken place.

2 The Model

There are $1 + N$ symmetric countries indexed by $i = 0, 1, \cdots, N$, where $N \geq 2$. Each country is populated by a continuum of workers who provide labor, the only factor of production, to a continuum of expected-profit-maximizing firms. Each firm produces a unique good, ω, and is characterized by its productivity, φ, which is drawn from a cumulative distribution function that strictly increases on $(0, \infty)$. A firm with productivity φ that

[13] These include Impullitti, Irarrazabal and Opromolla (2013), Handley (2014), and Handley and Limão (2015).

hires ℓ workers will produce $\varphi\ell$ units of its unique good.

Demand for good ω in country i depends on the realization of the firm's country-specific demand shock, $\gamma_i(\omega)$. Defining $p_i(\omega)$ as the price of good ω in country i, the CES demand for good ω in country i is

$$q_i(\omega) = \frac{\gamma_i(\omega) P_i^{\sigma} Q_i}{p_i(\omega)^{\sigma}}, \tag{1}$$

where

$$P_i = \left[\int_{\omega \in \Omega_i} \gamma_i(\omega) p_i(\omega)^{1-\sigma} d\omega \right]^{1/(1-\sigma)} \tag{2}$$

is the price index in country i and

$$Q_i = \left[\int_{\omega \in \Omega_i} \gamma_i(\omega)^{1/\sigma} q_i(\omega)^{\rho} d\omega \right]^{1/\rho} \tag{3}$$

is the aggregate consumption in country i, with Ω_i being the set of goods available in country i, and $\sigma > 1$ and $\rho = (\sigma - 1)/\sigma$.

A firm's country-specific demand shocks may be correlated and are drawn from a distribution that may vary among firms. Since the countries are symmetric, for any shock in the firm's own country, the shocks in all the other countries are exchangeable. That is, if the firm's own country is i, then for a given γ_i the shocks $(\gamma_0, \gamma_1, \cdots, \gamma_{i-1}, \gamma_{i+1}, \cdots, \gamma_N)$ and any of its permutations have the same distribution. A firm's draw of shock realizations is independent of other firms' draws.

Serving markets in countries other than its own requires a firm to incur a fixed export cost of $f_x > 0$ units of labor per market. The total fixed export cost is therefore nf_x, where $n \in \{0, 1, \cdots, N\}$ is the number of export destinations chosen by the firm. Due to an iceberg cost, the firm must ship $\tau \geq 1$ units of its good for each unit it sells abroad.

We focus on equilibria in which P_i, Q_i, and the wage are the same in each country. We therefore omit the subscript i in these variables. We also normalize the wage to unity.

Without loss of generality, we denote the home country as country zero and the foreign countries as countries $1, 2, \cdots, N$. Our analysis takes the point of view of the home country, but the results are identical for the other countries.

A firm makes its decisions in two stages. The first stage, which we refer to as the *setup stage*, takes place before the realization of the demand shocks. Here, the firm chooses a particular subset of foreign markets to serve and the number of workers to hire, ℓ. These choices cannot be changed later. Since all foreign countries are ex-ante identical, without loss of generality, we consider the firm's choice to be how many foreign countries to serve, n, and we assume that a firm exporting to $n > 0$ destinations serves the foreign countries indexed by $1, 2, \cdots, n$.

The second stage, which we refer to as the *allocation stage*, takes place after the realization of the demand shocks. The firm now chooses how to allocate its available output, $\varphi\ell$, across its home and chosen foreign markets. To earn the highest possible revenue from the output allocated to the ith market, the firm sets the price according to the inverse demand function obtained from (1),

$$p_i = \left(\frac{\gamma_i Q}{q_i}\right)^{1/\sigma} P \text{ for } i = 0, 1, \cdots, n. \tag{4}$$

Thus, incorporating the iceberg cost, the output constraint when q_i is allocated to the ith market is

$$q_0 + \tau \sum_{i=1}^{n} q_i = \varphi\ell. \tag{5}$$

Using (4), at the allocation stage the firm's revenue in the ith market is $p_i q_i = \gamma_i^{1/\sigma} q_i^{\rho} P Q^{1/\sigma}$ and its profit is

$$\left(\sum_{i=0}^{n} \gamma_i^{1/\sigma} q_i^{\rho}\right) P Q^{1/\sigma} - n f_x - \ell. \tag{6}$$

At the outset, therefore, the firm's expected profit is

$$\mathop{\mathrm{E}}_{\Gamma_n}\left(\sum_{i=0}^{n}\gamma_i^{1/\sigma}q_i^{\rho}\right)PQ^{1/\sigma}-nf_x-\ell, \tag{7}$$

where $\Gamma_n=(\gamma_0,\gamma_1,\cdots,\gamma_n)$ are the demand shocks at home and the n export destinations. The firm's objective in the two-stage decision problem is to maximize its expected profit (7) by choosing n and ℓ at the setup stage and choosing $q_0,q_1,\cdots,q_n$ subject to the output constraint (5) at the allocation stage.

2.1 Firm Optimization

A firm's optimization problem can be solved by iterating backward over the two decision stages outlined above.

Allocation Stage At this (second) stage, a firm takes the export destinations, the number of hired workers (and hence output), and its realized demand shocks as given. The q_i's that maximize the firm's profit (6) subject to the output constraint (5) are internal (since the isoelastic demand function has no saturation point) and satisfy

$$\begin{aligned} q_0 &= \frac{\gamma_0\varphi\ell}{S_n}; \\ q_i &= \frac{\gamma_i\varphi\ell}{S_n\tau^{\sigma}} \text{ for } i=1,2,\cdots,n, \end{aligned} \tag{8}$$

where

$$S_n\equiv\gamma_0+\tau^{1-\sigma}\sum_{i=1}^{n}\gamma_i$$

denotes the sum of the firm's realized country-specific shocks in its home market and n foreign markets after accounting for the iceberg cost (for a non-exporting firm, $S_0=\gamma_0$).

Accordingly, given its export destinations and number of workers hired, the firm's allocation of output to any particular market depends on the realized shocks in all of the firm's markets. As shown by (8), the output sold in one country increases with the size of the realized shock in that country and decreases with the size of the realized shock in all other countries. The ratio of the quantity sold in a foreign country to the shock realization in that country, q_i/γ_i, is the same for all foreign countries, and it equals $\tau^{-\sigma}$ times the ratio of the quantity sold in the home country to the shock realization in the home country, $q_0/(\gamma_0\tau^{\sigma})$.

The explanation is that at the allocation stage, maximizing profit is equivalent to maximizing total revenue. The latter requires that the marginal revenue in all of the firm's foreign markets (i.e., $(\gamma_i Q/q_i)^{1/\sigma}\rho P$ for $i = 1, 2, \cdots, n$) is the same and equals τ times the marginal revenue in the home market (i.e., $(\gamma_0 Q/q_0)^{1/\sigma}\rho P$). However, given the multiplicative nature of the demand shock and the CES demand, the inverse demand (4) depends on q_i and γ_i only through the multiplicative $(q_i/\gamma_i)^{-1/\sigma}$. This is therefore also true for the marginal revenue, from which it follows that q_i/γ_i must be the same for all foreign countries and equal to $q_0/(\gamma_0\tau^{\sigma})$ for the home country.

To determine the corresponding optimal prices, substitute the quantities (8) into the inverse demand function (4). This shows that the firm sets its price at home to

$$p_0 = \left(\frac{S_n Q}{\varphi \ell}\right)^{1/\sigma} P \tag{9}$$

and its price in foreign country $i = 1, 2, \cdots, n$ to $p_i = \tau p_0$. This price setting occurs because, in view of (1), the marginal revenue in every country, home or foreign, is equal to ρ times the price charged in that country. That is, the marginal revenue in a country depends only on the demand elasticity, which is the same in all countries, and the price charged in the country. Accordingly, as with the output sold, in every country the price depends on the realized shocks in all of the firm's markets. It follows from (9) that the price in a country increases with the realized shock in the country itself and in any of the other countries.

By substituting the quantities from (8) into (6), for a given n and ℓ, the firm's profit at the allocation stage is

$$(\varphi\ell)^{\rho} S_n^{1/\sigma} P Q^{1/\sigma} - n f_x - \ell. \tag{10}$$

Setup Stage At this (first) stage, using (10), the expected profit for a given n and ℓ is

$$(\varphi\ell)^{\rho} M(S_n)^{1/\sigma} P Q^{1/\sigma} - n f_x - \ell, \tag{11}$$

where

$$M(S_n) \equiv \left(\operatorname*{E}_{\Gamma_n} S_n^{1/\sigma}\right)^{\sigma}$$

is the generalized mean of degree $1/\sigma$ of S_n. A natural interpretation of $M(S_n)$ is that it is the certainty-equivalent sum of the firm's shocks in its home market and n foreign markets after accounting for the iceberg cost. How the distribution of S_n and the associated value of $M(S_n)$ change with n for different shock types plays a crucial role in our analysis.

If n were given, a firm with productivity φ would choose ℓ to maximize (11), that is,

$$\ell\,(n;\varphi) = \varphi^{\sigma-1}\rho^{\sigma} M(S_n) P^{\sigma} Q. \tag{12}$$

The profit would be

$$\pi(n;\varphi) = \varphi^{\sigma-1} P^{\sigma} Q \left[\rho^{\sigma-1} S_n^{1/\sigma} M(S_n)^{\rho} - \rho^{\upsilon\sigma} M(S_n)\right] - n f_x,$$

and the expected profit would be

$$\operatorname*{E}_{\Gamma_n} \pi\,(n;\varphi) = \varphi^{\sigma-1} k M(S_n) - n f_x, \tag{13}$$

where $k \equiv \rho^{\sigma-1} P^{\sigma} Q/\sigma$.[14] The firm's optimal number of export destinations is therefore[15]

$$n(\varphi) \equiv \underset{n}{\operatorname{argmax}} \underset{\Gamma_n}{\mathrm{E}} \pi(n; \varphi). \tag{14}$$

3 The Ex-Post Allocation Motive for Exporting

If a firm had to allocate its output across destinations *ex ante*, that is, prior to learning the realization of shocks, there would, in general, be a gap between the output the firm allocates to each destination and what it would have allocated had it known the realization of shocks. The firm would incur a loss from this output gap in each country, and the losses would be cumulative. However, when, as in our model, a firm can allocate output *ex post*, that is, after shocks are realized, what matters is only the gap between the firm's total output and the total output it would have chosen had it known the shock realizations. This is because, unless shocks are identical everywhere, the firm can partially offset a loss from low demand in one destination by shuffling output toward another destination with higher demand. Moreover, the more destinations the firm serves, the greater is its capacity for shuffling output across destinations in response to shock realizations. Hence, having more export destinations helps a firm minimize the losses from the output gaps that would

[14] Substituting $\ell(n; \varphi)$ from (12) into (8) and (9), we see that

$$q_0 = \frac{\gamma_0 \varphi^{\sigma} (\sigma - 1) k M(S_n)}{S_n};$$
$$q_i = \frac{\gamma_i \varphi^{\sigma} (\sigma - 1) k M(S_n)}{\tau^{\sigma} S_n} \text{ for } i = 1, 2, \cdots, n,$$

and

$$p_0 = \frac{S_n^{1/\sigma}}{\varphi \rho M(S_n)^{1/\sigma}}$$

with $p_i = \tau p_0$ for $i = 1, 2, \cdots, n$. Thus, the expected p_0 is $1/(\varphi\rho)$ and the expected p_i for $i = 1, 2, \cdots, n$ is $\tau/(\varphi\rho)$. These expected prices are exactly what the actual prices at home and abroad would be if, in contrast to our assumption, the firm could adjust its output after the shock realizations.

[15] The general-equilibrium values of P and Q are determined by the labor market clearing in each country. Since P and Q will play no significant role in the subsequent analysis, we omit the derivation of these values.

have occurred in each country had the firm allocated its output prior to learning the shock realizations.

We therefore decompose a firm's expected profit (13) into the *expected direct profit,* which is the expected profit if the firm commits to an allocation of output before to the realization of shocks, and the *expected ex-post allocation profit*, which is the additional expected profit that stems from the firm being able to allocate output ex post in response to shock realizations. The expected direct profit is

$$\varphi^{\sigma-1}k\left[M(\gamma_0)+n\tau^{1-\sigma}M(\gamma_1)\right]-nf_x, \tag{15}$$

where $\varphi^{\sigma-1}kM(\gamma_0)$ is the expected value of the revenue from the output sold in the home country less the cost of producing that output, and $\varphi^{\sigma-1}k\tau^{1-\sigma}M(\gamma_1)$, with $M(\gamma_1)\equiv\left(\underset{\gamma_1}{\mathrm{E}}\gamma_1^{1/\sigma}\right)^{\sigma}$, is the expected revenue from the output sold in each foreign country less the cost of producing that output.[16] The expected ex-post allocation profit is therefore (13) minus (15),

$$\varphi^{\sigma-1}k\{M(S_n)-\left[M(\gamma_0)+n\tau^{1-\sigma}M(\gamma_1)\right]\}. \tag{16}$$

The generalized mean of degree $1/\sigma$ of S_n is a concave function of shock realizations.[17] This implies that, unless the shock realizations in all foreign countries are always identical and proportional to the shock realization in the home country (what we call global shocks), for all $n>0$ the certainty-equivalent sum of the firm's shocks in all its markets, $M(S_n)$, exceeds the certainty-equivalent value of the firm's shock in its home market, $M(\gamma_0)$, plus the certainty-equivalent sum of the firm's shocks in its foreign markets, $n\tau^{1-\sigma}M(\gamma_1)$; that is, $M(S_n)>M(\gamma_0)+n\tau^{1-\sigma}M(\gamma_1)$ for any $n>0$. Hence, the expected ex-post allocation

[16] Since each foreign country has the same shock distribution, $\left(\underset{\gamma_i}{\mathrm{E}}\gamma_i^{1/\sigma}\right)^{\sigma}$ is the same for $i=1,2,\cdots,n$.

[17] While we are mostly concerned with how $M(S_n)$ depends on n, here and in a few more places we consider the effect of the shock distributions on $M(S_n)$ for a given n.

profit (16) is positive for any $n > 0$. This reflects that at the allocation stage a firm increases its profit by exploiting the differences in shock realizations when distributing its output among countries.

The upshot is that a firm's ability to allocate output ex post increases its expected profit by giving it a more diverse customer base. That is, even if exporting to more countries only increased the diversity of a firm's customer base without increasing the size of the customer base, the firm's expected profit would still increase because of the firm's ability to divert output from markets with low demand toward markets with high demand. Thus, besides the standard motive for exporting (reaching a larger customer base), the expected gain from the ability to allocate output in response to shock realizations constitutes an additional motive for exporting, which we refer to as the *ex-post allocation motive for exporting*.

4 Firms' Export Behavior

A firm's gain in expected profit from exporting to $n+1$ rather than n countries is $\underset{\Gamma_{n+1}}{\mathrm{E}}\,\pi(n+1;\varphi) - \underset{\Gamma_n}{\mathrm{E}}\pi(n;\varphi) = \varphi^{\sigma-1}k\left[M(S_{n+1}) - M(S_n)\right] - f_x$. Since $\varphi^{\sigma-1}k\left[M(S_{n+1}) - M(S_n)\right]$ is proportional to $\varphi^{\sigma-1}$ and f_x is independent of φ, the optimal number of export destinations given by (14) weakly increases in φ. Thus, there exist endogenous export cutoffs, $\varphi_1 \leq \varphi_2 \leq \cdots \leq \varphi_N$, such that firms with $\varphi < \varphi_n$ export to fewer than n foreign countries, while firms with $\varphi \geq \varphi_n$ export to at least n foreign countries.[18] The fixed cost of exporting entails that firms with low productivity do not export. This is tantamount to $0 < \varphi_1$. Also, very productive firms export to all foreign countries. It follows that $\varphi_N < \infty$.

The export solution is bang-bang if $\varphi_1 = \varphi_N$ (i.e., $\varphi_1 = \varphi_2 = \cdots = \varphi_N$) and not

[18] If $\varphi_n = \varphi_{n+1} = \cdots = \varphi_{n+z} < \varphi_{n+z+1}$ for some $z \geq 1$, then no firm exports to exactly $n, n+1, \cdots, n+z-1$ foreign countries, while firms whose productivity falls in the interval $[\varphi_n, \varphi_{n+z+1})$ export to exactly $n+z$ foreign countries.

bang-bang if $\varphi_1 < \varphi_N$. As we show in the next section, even with symmetric countries, the ex-post allocation motive for exporting may rule out a bang-bang export solution and cause a firm to export to some but not all foreign destinations. Thus, the ability to allocate output in response to shock realizations may overturn the result in a standard Melitz framework that firms either do not export or export everywhere.

The optimal number of export destinations depends on how the expected profit function, $\underset{\Gamma_n}{\mathrm{E}}\pi(n;\varphi)$, varies with n, and therefore, in view of (13), depends on the certainty-equivalent sum of the firm's shocks, $M(S_n)$, as a function of n. Specifically, a necessary and sufficient condition for the export solution to *not* be bang-bang is that for some firm productivity there exists an n for which $\underset{\Gamma_n}{\mathrm{E}}\pi(n;\varphi)$ exceeds both $\underset{\gamma_0}{\mathrm{E}}\pi(0;\varphi)$ and $\underset{\Gamma_N}{\mathrm{E}}\pi(N;\varphi)$. This is equivalent to the existence of an n, $0 < n < N$, for which

$$\frac{M(S_n) - M(\gamma_0)}{n} > \frac{M(S_N) - M(\gamma_0)}{N}. \tag{17}$$

That is, for some n, $0 < n < N$, the gain per export destination in the certainty-equivalent sum of the shocks from exporting to n destinations exceeds that obtainable from exporting to all N destinations.

5 Overturning the Bang-Bang Solution

In this section we prove that the ability to allocate output ex post can overturn the result of a bang-bang solution even when countries are symmetric. To facilitate the analysis, we assume that the economy has three sectors, with each sector characterized by a particular shock correlation across markets. The sectors are otherwise similar, and the demand for goods in each sector continues to be given by (1).[19] The three sectors are:

[19] The price index and aggregate consumption are still given by (2) and (3), so everything said until now remains accurate.

Sector with Global Shocks Shock realizations are the same in all foreign countries and proportional to the shock realization in the home country. That is, $\gamma_1 = \gamma_2 = \cdots = \gamma_N = \nu\gamma_0$, where $\nu > 0$, in all realizations. Using the superscript g to indicate the sector with global shocks, $S_n^g = (1 + n\tau^{1-\sigma}v)\gamma_0$. Global shocks may emerge for goods whose popularity is determined by wars, worldwide pandemics, or international fads and fashions.

Sector with Common Foreign Shocks Shock realizations are the same in all foreign countries and independent of the shock realization in the home country. That is, $\gamma_1 = \gamma_2 = \cdots = \gamma_N$ and independent of γ_0 in all realizations. Using the superscript c to indicate the sector with common foreign shocks, $S_n^c = \gamma_0 + n\tau^{1-\sigma}\gamma_1$. Common foreign shocks may arise when there are product-specific home or foreign biases of varying and uncertain magnitudes.

Sector with Idiosyncratic Foreign Shocks Shock realizations are independent across the home country and all foreign countries. That is, $\gamma_0, \gamma_1, \cdots, \gamma_N$ are independent in all realizations. Idiosyncratic foreign shocks could arise because consumers in different countries are exposed to different cultural and environmental forces and therefore undergo different taste changes that firms cannot predict.

In the following subsections we determine the nature of the export solution in each of the three sectors.

5.1 Sector with Global Shocks

When shock realizations are identical across all foreign destinations and proportional to the shock realization at home, a bang-bang solution emerges.[20]

[20] The proofs of the propositions are in the appendix.

Proposition 1 *If shocks are global, then the export solution is bang-bang. That is, the export cutoffs satisfy* $\varphi_1^g = \varphi_N^g$.

Since global shock realizations are equally favorable or unfavorable everywhere (taking ν into account), the optimal allocation of the firm's output is identical for every shock realization. Specifically, the firm always sells the same quantity in each foreign country and τ^σ/ν times that quantity in the home country. Because it is not possible for the firm to gain by allocating its output differently across destinations for different shock realizations, the expected ex-post allocation profit is zero for any n.

As a consequence, when a firm chooses the number of export destinations, it, in effect, does so by considering each destination in isolation. Accordingly, if it is worthwhile to export to one foreign country, it is worthwhile to export to all foreign countries and a bang-bang solution ensues. In particular, if a firm's productivity is below (above) the threshold value given by the cutoff for starting to export, φ_1^g, then its expected gain in profit from exporting to one additional foreign country is negative (positive) for all $n > 0$, and the firm therefore does not export (exports to all foreign countries).

5.2 Sector with Common Foreign Shocks

When shock realizations are identical across all foreign destinations and independent of the home shock realization, not only is the bang-bang solution overturned, but any particular number of export destinations will be optimal for some firm productivity.

Proposition 2 *If foreign shocks are common, then the export solution is not bang-bang. Furthermore, there are firms exporting to* $0, 1, \cdots, N$ *foreign countries. That is, the export cutoffs satisfy* $\varphi_1^c < \varphi_2^c < \cdots < \varphi_N^c$.

The proof shows that when foreign shocks are common, the gain in the expected ex-post allocation profit from adding another export destination decreases with the number of export destinations. Condition (17) is then satisfied so that the optimal export solution is not bang-bang. Not only that, but since a firm's expected profit increases with firm productivity, for any n, $0 < n < N$, there is an interval of productivity, such that firms with productivity in that interval gain from adding each of the first n export destinations but lose from adding the $(n+1)$th export destination. However, a firm with a higher productivity gains from adding the $(n+1)$th export destination. Given that firms with low productivity do not export and firms with high productivity export to all foreign countries, it follows that there are firms that export to exactly n destinations for any $n = 0, 1, \cdots, N$.

To understand why the gain in the expected ex-post allocation profit from exporting to one more destination decreases in n, consider the consequences of exporting for the allocation of a firm's output between the home and foreign markets. If the firm serves only the home market, it is forced to sell all its output at home regardless of the shock realization in the home country. Adding one export destination makes it possible for the firm to allocate its output between the two markets in response to shock realizations. In particular, the more (less) favorable the shock realization in the home country relative to the shock realization in the foreign country, the more (less) output the firm allocates to the home country. This helps mitigate the loss associated with the uncertain demand at home. Adding more export destinations further facilitates the shuffling of the firm's output between the home and foreign markets in response to shock realizations. However, as the number of export destinations grows, and hence more (less) output is allocated to the home country and away from the foreign countries in the case of a relatively favorable (unfavorable) home-country shock realization, the benefit diminishes from any further shuffling of output facilitated by another export destination.

5.3 Sector with Idiosyncratic Foreign Shocks

When shock realizations are independent across the home and all foreign destinations, the export solution is not necessarily bang-bang. Using the superscript d to indicate the sector with idiosyncratic foreign shocks, the following is true:

Proposition 3 *If foreign shocks are idiosyncratic, then the export solution may be either bang-bang or not bang-bang. That is, the export cutoffs satisfy either $\varphi_1^d = \varphi_N^d$ or $\varphi_1^d < \varphi_N^d$.*

The key difference between common and idiosyncratic foreign shocks is that with common foreign shocks all foreign markets have equally favorable shock realizations and are therefore allocated the same amount of output. Not so in the case of idiosyncratic foreign shocks, in which shock realizations differ across foreign markets and more output is allocated toward the foreign markets with relatively favorable shock realizations. Thus, in the sector with idiosyncratic foreign shocks, a firm benefits not only from its ability to determine how much to export, but also from its ability to allocate relatively more of that export toward the foreign markets with relatively favorable shock realizations.

To elucidate the value to a firm of being able to allocate its exports between its foreign markets based on the shock realizations in these markets, consider a scenario in which a firm is free to shuffle output across foreign markets but not between the home and foreign markets. Then, for each export destination, the expected value of the revenue from the output sold at that destination less the cost of producing that output is proportional to the certainty-equivalent value of the average shock in the export destinations, that is, to the generalized mean, $\left[\underset{\Gamma_n}{\mathrm{E}}\left(\sum_{i=1}^{n}\gamma_i\right)^{1/\sigma}\right]^{\sigma}/n$. Because the foreign shocks are independent of one another, the distribution of the average shock in the export destinations becomes less spread out as the number of export destinations increases. As the generalized mean of the average shocks at the export destinations is a concave function of the average shocks, the certainty-equivalent value of the average shock at an export destination and therefore the expected value of the revenue from the output sold at that destination less the cost of producing that output increases with the number of export destinations. Hence the gain in the expected ex-post allocation profit, which stems from a firm's ability to allocate a given total export among its export destinations, also increases with the number of export destinations.

The proof of Proposition 3 considers the case of $N = 2$ and shows that the increase in the certainty-equivalent sum of the shocks from adding the first export destination may either exceed (Condition (17) is satisfied) or fall short of (Condition (17) is not satisfied) the increase in the certainty-equivalent sum of the shocks from adding the second export destination. Consequently, the export solution may or may not be bang-bang.

6 The Importance of Shock Correlations

In this section we illuminate the mechanism underlying the ex-post allocation motive for exporting by exploring how the shock correlations across markets in the different sectors influence a firm's incentive to export and the associated pattern of cutoffs for exporting to a particular number of foreign countries. The crucial insight is that the benefit to a firm from being able to shuffle output from destinations with relatively unfavorable shock realizations to destinations with relatively favorable shock realizations is negatively related to the degree to which shock realizations are correlated across destinations. Therefore, the export cutoffs increase with the correlation of shock realizations across destinations.

To properly compare the sectors, we assume that in each sector firms draw all their country-specific shock realizations from the same unconditional shock distribution. Thus, in the sector with global shocks, a firm draws one shock realization from this distribution, which applies to both the firm's home market (proportionally) and to all of its foreign markets; in the sector with common foreign shocks, a firm draws two independent realizations from this distribution, one applying to its home market and one applying to all of its foreign markets; and in the sector with idiosyncratic foreign shocks, a firm exporting to n foreign markets draws $n+1$ independent realizations from this distribution, each applying to one country.

While highly stylized, comparing firms' export behavior in this way allows us to home in on the impact of shock correlations across markets on the ex-post allocation motive for exporting. Indeed, since all shock realizations in the three sectors are drawn from the same distribution, the unconditional shock distribution in each country, home or foreign, is identical in all three sectors. Any difference in firms' export behavior across sectors is therefore attributable to the ex-post allocation motive for exporting and driven by how the value of the option to allocate output in response to shock realizations depends on the correlation of shocks across markets. We now juxtapose the export behavior as expressed in the export cutoffs in the three sectors.

Proposition 4 *The export cutoffs satisfy* $\varphi_1^d \leq \varphi_1^c$ *and* $\varphi_n^d < \varphi_n^c$ *for* $n > 1$*; and* $\varphi_N^c < \varphi_1^g$.

Accordingly, for a given firm productivity, firms with idiosyncratic foreign shocks export to at least as many foreign countries as firms with common foreign shocks, which themselves export to at least as many foreign countries as firms with global shocks. We now elucidate the mechanism, explaining first the difference in export cutoffs between the sectors with idiosyncratic and common foreign shocks, and then the difference in export cutoffs between the sectors with common foreign and global shocks.

Idiosyncratic versus Common Foreign Shocks The cutoffs for firms starting to export is either the same in the two sectors, $\varphi_1^d = \varphi_1^c$, or lower in the sector with idiosyncratic foreign shocks than in the sector with common foreign shocks, $\varphi_1^d < \varphi_1^c$. To understand when the cutoffs are the same, observe that for a firm exporting to a single foreign country, it is immaterial whether the foreign uncertainty is due to idiosyncratic or common foreign shocks. Therefore, if, in the sector with idiosyncratic foreign shocks, some firms export to a single foreign country ($\varphi_1^d < \varphi_2^d$), then the cutoff for starting to export is the same as in the sector with common foreign shocks, $\varphi_1^d = \varphi_1^c$.[21]

To understand when the cutoff is lower in the sector with idiosyncratic foreign shocks, observe that if no firm with idiosyncratic foreign shocks exports to just one foreign country, then for some $\underline{n} \geq 2$ foreign countries $1/\underline{n}$ of the expected ex-post allocation profit from exporting exceeds that from exporting to just one foreign country. This makes it beneficial to start exporting at a lower firm productivity. Therefore, if, in the sector with idiosyncratic foreign shocks, no firm exports to just one foreign country ($\varphi_1^d = \varphi_2^d$), then the cutoff for starting to export is lower in the sector with idiosyncratic foreign shocks than in the sector with common foreign shocks, $\varphi_1^d < \varphi_1^c$.[22]

Turning to the cutoff for exporting to at least $n > 1$ foreign countries, it is always lower in the sector with idiosyncratic foreign shocks than in the sector with common foreign shocks, $\varphi_n^d < \varphi_n^c$ for $n > 1$. The reason is that for a firm exporting to more than one foreign country, the gain in the expected ex-post allocation profit from exporting to one additional foreign country is higher with idiosyncratic foreign shocks, as a firm can benefit from shuffling output between the foreign countries, than with common foreign shocks, as there is no such benefit.

Common Foreign versus Global Shocks In contrast to the sectors with idiosyncratic or common foreign shocks, in the sector with global shocks there is no expected ex-post allocation profit. Consequently, in the sector with common foreign shocks (and in the sector with idiosyncratic foreign shocks) the increase in expected profit from adding an

[21] With the distribution of shocks in the second example in the proof of Proposition 3, there can be firms exporting to a single foreign country.

[22] With the distribution of shocks in the first example in the proof of Proposition 3, if $N = 3$ instead of $N = 2$, no firm will export to just one foreign country while some firms will export to two foreign countries.

export destination always exceeds what can be obtained in the sector with global shocks from adding even the first export destination. The implication is that there exists a range of productivity for which firms in the sector with common foreign shocks export, while firms within the same range in the sector with global shocks do not export at all. That is, $\varphi_N^c < \varphi_1^g$.

Together with the results from comparing the sectors with idiosyncratic and common foreign shocks, it follows that the cutoff for exporting to all N foreign countries is smallest with idiosyncratic foreign shocks, higher with common foreign shocks, and highest with global shocks, $\varphi_N^d < \varphi_N^c < \varphi_N^g$. Accordingly, there are firm productivities for which firms with idiosyncratic foreign shocks export to all foreign countries, firms with common foreign shocks export to some but not all foreign countries, and firms with global shocks do not export at all.

7 Conclusion

This paper has developed some of the theoretical implications of uncertainty in a standard trade model with risk-neutral firms and symmetric countries. We introduced uncertainty as firm-destination-specific demand shocks. Motivated by the empirical evidence, we assumed that firms must commit to their total output, but not the allocation of that output across destinations, before the shocks are realized. Firms also choose their export destinations before the uncertainty is resolved, because time is needed to set up the infrastructure necessary for selling in a particular market. Consequently, the distribution of shocks plays an important role in determining a firm's optimal number of export destinations.

Because of the irreversibility of the production decision, risk-neutral firms may choose to export not only to increase their total customer base but also to gain from the diversity of sources of demand, that is, from the imperfect correlation of demand at different locations.

We refer to this as the ex-post allocation motive for exporting. In particular, exporters can increase their realized (and hence expected) profit by directing more of their output to destinations with relatively favorable shock realizations and less to destinations with relatively unfavorable shock realizations. The correlations of the shocks in the different countries therefore affect how a firm's expected profit varies with the number of export destinations and hence the optimal number of export destinations.

We showed that in some scenarios the gain in a firm's expected profit from an additional export market decreases with the number of markets to which it exports. In these scenarios, a firm may find it worthwhile to incur the fixed export cost to some but not all destinations.

The optimal number of export destinations increases with the firm's productivity. This result was obtained even though countries are ex-ante completely symmetric: they do not differ in size, demand structure, export cost, or in any other way, implying no hierarchical order in the desirability of the different export destinations before the shock realizations. To elaborate, while our model can explain why firms may optimally decide to serve some, but not all, ex-ante identical destinations, it has no predictive power regarding which countries will be served. Moreover, we showed that for a given productivity, a firm exports to at least as many foreign destinations when the foreign shocks are idiosyncratic (i.e., uncorrelated and also uncorrelated with the home shock) as when the foreign shocks are common (i.e., perfectly correlated but uncorrelated with the home shock) and exports to at least as many foreign destinations when the foreign shocks are common as when all shocks are global (i.e., perfectly correlated).

The timing of the resolution of uncertainty is important. However, besides the shocks highlighted in our paper, firms likely face additional uncertainty, some of which is resolved prior to production and some of which is resolved only after output has reached its final destination. Including these additional sources of uncertainty in the analysis might lead to new insights, but would not overturn the basic insights in this paper. Indeed, so long as some uncertainty is resolved after production but before the output has reached its final destination, even risk-neutral firms can benefit from having a diverse customer base.

While our analysis has abstracted from the ability of firms to hold inventories, such an ability would provide firms with another means of shuffling a given output after shock realizations become known. Indeed, inventories can be thought of as an additional "destination" for firms' output, thus giving firms the ability to allocate their output across time as well as across space, that is, countries, as in our paper. Since inventories are costly, firms would still benefit from the ability to shuffle their output across space, and therefore the gist of our results would remain unchanged.

References

Ahn, J., McQuoid, A.F., 2017. Capacity constrained exporters: Identifying increasing marginal cost. Economic Inquiry 55, 1175-1191.

Albornoz, F., Pardo, H.F., Corcos, G., Ornelas, E., 2012. Sequential exporting. Journal of International Economics 88, 17-31.

Almunia, M., Antràs, P., Lopez-Rodriguez, D., Morales, E. 2021. Venting out: Exports during a domestic slump. American Economic Review 111, 3611-3662.

Amiti, M., Davis, D.R., 2011. Trade, firms, and wages: Theory and evidence. The Review of Economic Studies 79, 1-36.

Arkolakis, C., 2010. Market penetration costs and the new consumers margin in international trade. Journal of Political Economics 118, 1151-1199.

Berman, N., Berthou, A., Héricourt, J., 2015. Export dynamics and sales at home. Journal of International Economics 96, 298-310.

Bernard, A.B., Jensen, J.B., Redding, S.J., Schott, P.K., 2007. Firms in international trade. Journal of Economic Perspectives 21, 105-130.

Blum, B.S., Claro, S., Horstmann, I.J., 2013. Occasional and perennial exporters. Journal of International Economics 90, 65-74.

Chaney, T., 2008. Distorted gravity: the intensive and extensive margins of international trade. The American Economic Review 98, 1707-1721.

Danziger, E., Danziger L., forthcoming. A minimum wage may increase exports and firm size even with a competitive labor market. Research in Labor Economics.

Eaton, J., Eslava, M., Kugler, M., Tybout, J., 2008. The margins of entry into export markets: Evidence from Colombia, 2008. In Helpman, E., Marin, D., Verdier, T. (eds.), The organization of firms in a global economy. Harvard University Press, 231-272.

Eaton, J., Kortum, S., Kramarz, F., 2004. Dissecting trade: Firms, industries, and export destinations. American Economic Review 94, 150-154.

Eaton, J., Kortum, S., Kramarz, F., 2011. An anatomy of international trade: Evidence from French firms. Econometrica 79, 1453-1498.

Esposito, F., 2022. Demand risk and diversification through trade. Journal of International Economics 135, 1-25.

Evans, C.L., Harrigan, J., 2005. Distance, time, and specialization: Lean retailing in general equilibrium. American Economic Review 95, 292-313.

Fillat, J.L., Garetto, S., 2015. Risk, returns, and multinational production. The Quarterly Journal of Economics 130, 2027-2073.

Fillat, J.L., Garetto, S., Oldenski, L., 2015. Diversification, cost structure, and the risk premium of multinational corporations. Journal of International Economics 96, 37-54.

Haddad, M., Lim, J.J., Pancaro, C., Saborowski, C., 2013. Trade openness reduces growth volatility when countries are well diversified. Canadian Journal of Economics 46, 765-790.

Handley, K., 2014. Exporting under trade policy uncertainty: Theory and evidence. Journal of International Economics 94, 50-66.

Handley, K., Limão, N., 2015. Trade and investment under policy uncertainty: Theory and firm evidence. American Economic Journal: Economic Policy 7, 189-222.

Helpman, E., Melitz, M., Rubinstein, Y., 2008. Estimating trade flows: Trading partners and trading volumes. The Quarterly Journal of Economics 123, 441-487.

Hummels, D.L., Schaur, G., 2010. Hedging price volatility using fast transport. Journal of International Economics 82, 15-25.

Impullitti, G., Irarrazabal, A.A., Opromolla, L.D., 2013. A theory of entry into and exit from export markets. Journal of International Economics 90, 75-90.

Melitz, M.J., 2003. The impact of trade on intra-industry reallocations and aggregate industry productivity. Econometrica 71, 1695-1725.

Munch, J.R. and Nguyen, D.X., 2014. Decomposing firm-level sales variation. Journal of Economic Behavior and Organization 106, 317-334.

Nguyen, D.X., 2012. Demand uncertainty: Exporting delays and exporting failures. Journal of International Economics 86, 336-44.

Soderbery, A., 2014. Market size, structure, and access: Trade with capacity constraints. European Economic Review 70, 276-298.

Vannoorenberghe, G., 2012. Firm-level volatility and exports. Journal of International Economics 86, 57-67.

Vannoorenberghe, G., Wang, Z., Yu, Z., 2016. Volatility and diversification of exports: Firm-level theory and evidence. European Economic Review 89, 216-247.

Appendix

Proof of Proposition 1

In the sector with global shocks, the certainty-equivalent sum of the shocks is $M(S_n^g) = (1 + n\tau^{1-\sigma}\nu) \underset{\gamma_0}{\mathrm{E}} \left(\gamma_0^{1/\sigma}\right)^\sigma$ and hence linear in n. Therefore, $M(S_n^g) - M(S_{n-1}^g)$ is independent of n so that the gain in the expected profit from adding another export destination, $\varphi^{\sigma-1} k \left[M(S_n^g) - M(S_{n-1}^g)\right] - f_x$, is the same for all n. It follows that

$$\varphi_1^g = \varphi_2^g = \cdots = \varphi_N^g = \left\{ \frac{f_x}{k\left[M(S_1^g) - M(S_0^g)\right]} \right\}^{1/(\sigma-1)}.$$

That is, $\varphi_1^g = \varphi_N^g$ and the export solution is bang-bang. Firms with $\varphi < \varphi_1^g$ do not export, and firms with $\varphi \geq \varphi_1^g$ export to all foreign countries. ■

Proof of Proposition 2

In the sector with common foreign shocks, the certainty-equivalent sum of the shocks is $M(S_n^c) = \left[\underset{\Gamma_1}{\mathrm{E}}(\gamma_0 + n\tau^{1-\sigma}\gamma_1)^{1/\sigma}\right]^\sigma$. First, we show

Lemma 1 The series $\{M(S_n^c)\}_{n=0}^N$ is concave in n.

Proof Considering n as a continuous variable, we have that

$$\begin{aligned} \frac{\partial^2 M(S_n^c)}{\partial n^2} &= (\sigma - 1) M(S_n^c)^{(\sigma-2)/\sigma} \left[\frac{\partial M(S_n^c)^{1/\sigma}}{\partial n}\right]^2 + M(S_n^c)^{(\sigma-1)/\sigma} \frac{\partial^2 M(S_n^c)^{1/\sigma}}{\partial n^2} \\ &= M(S_n^c)^{(\sigma-2)/\sigma} \left\{ (\sigma - 1) \left[\frac{\partial M(S_n^c)^{1/\sigma}}{\partial n}\right]^2 + M(S_n^c)^{1/\sigma} \frac{\partial^2 M(S_n^c)^{1/\sigma}}{\partial n^2} \right\}. \end{aligned}$$

As $M(S_n^c) > 0$, it suffices to show that

$$(\sigma - 1) \left[\frac{\partial M(S_n^c)^{1/\sigma}}{\partial n}\right]^2 + M(S_n^c)^{1/\sigma} \frac{\partial^2 M(S_n^c)^{1/\sigma}}{\partial n^2} \tag{18}$$

is negative. Since

$$
\begin{aligned}
\frac{\partial M(S_n^c)^{1/\sigma}}{\partial n} &= \frac{\tau^{1-\sigma}}{\sigma} \mathop{\mathrm{E}}_{\Gamma_1}\left[\gamma_1 S_n^{(1-\sigma)/\sigma}\right], \\
\frac{\partial^2 M(S_n^c)^{1/\sigma}}{\partial n^2} &= \frac{(1-\sigma)\tau^{2-2\sigma}}{\sigma^2} \mathop{\mathrm{E}}_{\Gamma_1}\left[\gamma_1^2 S_n^{(1-2\sigma)/\sigma}\right],
\end{aligned}
$$

(18) has the same sign as

$$
\mathop{\mathrm{E}}_{\Gamma_1}\left[\gamma_1 S_n^{(1-\sigma)/\sigma}\right] \mathop{\mathrm{E}}_{\Gamma_1}\left[\gamma_1 S_n^{(1-\sigma)/\sigma}\right] - \mathop{\mathrm{E}}_{\Gamma_1} S_n^{1/\sigma} \mathop{\mathrm{E}}_{\Gamma_1}\left[\gamma_1^2 S_n^{(1-2\sigma)/\sigma}\right].
$$

Using primes for realizations in the second expectation in each term, this can be written as

$$
\begin{aligned}
&\mathop{\mathrm{E}}_{\Gamma_1}\mathop{\mathrm{E}}_{\Gamma_1'}\left[\gamma_1 S_n^{(1-\sigma)/\sigma}\gamma_1' S_n'^{(1-\sigma)/\sigma} - S_n^{1/\sigma}\gamma_1'^2 S_n'^{(1-2\sigma)/\sigma}\right] \\
&= \mathop{\mathrm{E}}_{\Gamma_1}\mathop{\mathrm{E}}_{\Gamma_1'}\left[S_n^{(1-2\sigma)/\sigma} S_n'^{(1-2\sigma)/\sigma}\left(\gamma_1 S_n \gamma_1' S_n' - S_n^2\gamma_1'^2\right)\right],
\end{aligned}
$$

or as

$$
\begin{aligned}
&\tfrac{1}{2}\mathop{\mathrm{E}}_{\Gamma_1}\mathop{\mathrm{E}}_{\Gamma_1'}\left[S_n^{(1-2\sigma)/\sigma} S_n'^{(1-2\sigma)/\sigma}\left(\gamma_1 S_n \gamma_1' S_n' - S_n^2\gamma_1'^2\right)\right] \\
&+ \tfrac{1}{2}\mathop{\mathrm{E}}_{\Gamma_1}\mathop{\mathrm{E}}_{\Gamma_1'}\left[S_n'^{(1-2\sigma)/\sigma} S_n^{(1-2\sigma)/\sigma}\left(\gamma_1' S_n' \gamma_1 S_n - \gamma_1^2 S_n'^2\right)\right],
\end{aligned}
$$

which follows from the fact that the expressions in the two double expectations are identical except for the switching of the primes. Since each pair of realizations Γ_1 and Γ_1' in the first double expectation has the same likelihood as the pair of realizations Γ_1' and Γ_1 in the second double expectation, the last expression becomes

$$
\begin{aligned}
&\tfrac{1}{2}\mathop{\mathrm{E}}_{\Gamma_1}\mathop{\mathrm{E}}_{\Gamma_1'}\left[S_n^{(1-2\sigma)/\sigma} S_n'^{(1-2\sigma)/\sigma}\left(2\gamma_1 S_n \gamma_1' S_n' - S_n^2\gamma_1'^2 - \gamma_1^2 S_n'^2\right)\right] \\
&= -\tfrac{1}{2}\mathop{\mathrm{E}}_{\Gamma_1}\mathop{\mathrm{E}}_{\Gamma_1'}\left[S_n^{(1-2\sigma)/\sigma} S_n'^{(1-2\sigma)/\sigma}\left(S_n\gamma_1' - \gamma_1 S_n'\right)^2\right]. \qquad (19)
\end{aligned}
$$

Since $S_n\gamma_1' - \gamma_1 S_n' = \gamma_0\gamma_1' - \gamma_1\gamma_0'$ and the home and foreign shock realizations are generally different, it follows that (19) and hence (18) is negative. Thus, the series $\{M(S_n^c)\}_{n=0}^N$ is concave in n. $\square$

Next, the fact that the series $\{M(S_n^c)\}_{n=0}^N$ is increasing and concave in n implies that $M(S_n^c) - M(S_{n-1}^c)$ decreases with n. In view of (13), it follows that the gain in the expected profit from adding the nth export destination, $\varphi^{\sigma-1}k\left[M(S_n^c) - M(S_{n-1}^c)\right] - f_x$, decreases with n. It also increases with φ. Accordingly, the cutoff for exporting to at least n destinations is determined by

$$\begin{aligned} (\varphi_n^c)^{\sigma-1}\, k\left[M(S_n^c) - M(S_{n-1}^c)\right] &= f_x \\ \Leftrightarrow \qquad \varphi_n^c &= \left\{\frac{f_x}{k\left[M(S_n^c) - M(S_{n-1}^c)\right]}\right\}^{1/(\sigma-1)}, \end{aligned} \tag{20}$$

from which it follows that $0 < \varphi_1^c < \cdots < \varphi_N^c$. Consequently, the export solution is not bang-bang and there are firms exporting to $n = 0, 1, \cdots, N$ foreign countries. $\blacksquare$

Proof of Proposition 3

In the sector with idiosyncratic foreign shocks, the certainty-equivalent sum of the shocks is

$$M(S_n^d) = \left[\underset{\Gamma}{\mathrm{E}}\left(\gamma_0 + \tau^{1-\sigma}\sum_{i=1}^{n}\gamma_i\right)^{1/\sigma}\right]^{\sigma}.$$

Suppose that all shocks are independently and uniformly distributed on $\left[\frac{1}{2}, \frac{3}{2}\right]$, $\sigma = 2$, and $N = 2$.

Example 1 If $\tau = 1$, then $M(S_1^d) - M(S_0^d) = 0.0006$ and $\left[M(S_2^d) - M(S_0^d)\right]/2 = 0.0007$. Condition (17) is not satisfied for $n = 1$ and the export solution is bang-bang.

Example 2 If $\tau = 0.8$, then $M(S_1^d) - M(S_0^d) = 0.0011$ and $\left[M(S_2^d) - M(S_0^d)\right]/2 = 0.0008$. Condition (17) is satisfied for $n = 1$ and the export solution is not bang-bang. $\blacksquare$

Proof of Proposition 4

We divide the proof into part A, which proves that $\varphi_1^d \leq \varphi_1^c$ and $\varphi_n^d < \varphi_n^c$ for $n > 1$, and part B, which proves that $\varphi_N^c < \varphi_1^g$.

Part A: Proof that $\varphi_1^d \leq \varphi_1^c$ and $\varphi_n^d < \varphi_n^c$ for $n > 1$.

We begin by showing

Lemma 2 $M(S^c_{n+1}) - M(S^c_n) < M(S^d_{n+1}) - M(S^d_n)$ for $n > 0$.

Proof For $n > 0$, $M(S^c_{n+1})^{1/\sigma} < M(S^d_{n+1})^{1/\sigma}$, from which it follows that

$$\begin{array}{rrcl} & M(S^c_{n+1})^{1/\sigma} - M(S^c_n)^{1/\sigma} & < & M(S^d_{n+1})^{1/\sigma} - M(S^d_n)^{1/\sigma} \\ \Rightarrow & M(S^c_{n+1}) - M(S^c_n) & < & M(S^d_{n+1}) - M(S^d_n). \end{array}$$

It therefore suffices to show that $M(S^c_{n+1})^{1/\sigma} - M(S^c_n)^{1/\sigma} < M(S^d_{n+1})^{1/\sigma} - M(S^d_n)^{1/\sigma}$ for $n > 0$. Define

$$\begin{aligned} \hat{S}^c_{n+\mu} &= \gamma_0 + (n+\mu)\tau^{1-\sigma}\gamma_1, \\ \hat{S}^d_{n+\mu} &= \gamma_0 + (n+\mu)\tau^{1-\sigma}a(n,\mu), \end{aligned}$$

where

$$a(n,\mu) \equiv \frac{\sum_{i=1}^n \gamma_i + \mu\gamma_{n+1}}{n+\mu},$$

and $\mu \in [0,1]$. Hence, $\hat{S}^j_n = S^j_n$ and $\hat{S}^j_{n+1} = S^j_{n+1}$ for $j = c, d$. To prove Lemma 2, we show that the derivative of $M(\hat{S}^d_{n+\mu})^{1/\sigma} - M(\hat{S}^c_{n+\mu})^{1/\sigma}$ with respect to μ is positive. This derivative has the same sign as

$$\begin{aligned} & (n+\mu)\left[M(\hat{S}^d_{n+\mu})^{1/\sigma} - M(\hat{S}^c_{n+\mu})^{1/\sigma}\right] \\ & + \tau^{1-\sigma} \underset{\Gamma_{n+1}}{\mathrm{E}} \left\{ \left[\frac{\gamma_0}{n+\mu} + \tau^{1-\sigma}a(n,\mu)\right]^{-\rho} \left(n\gamma_{n+1} - \sum_{i=1}^n \gamma_i\right)\right\} \\ & - \underset{\Gamma_{n+1}}{\mathrm{E}} \left\{ \left[\frac{\gamma_0}{n+\mu} + \tau^{1-\sigma}a(n,\mu)\right]^{-\rho} \gamma_0 \right\} \\ & + \underset{\Gamma_1}{\mathrm{E}} \left[\left(\frac{\gamma_0}{n+\mu} + \tau^{1-\sigma}\gamma_1\right)^{-\rho} \gamma_0 \right]. \end{aligned}$$

The term on the first line is zero for $n = 0$ and positive for $n > 0$. The term on the second line captures the effect of an increase in μ through $a(n,\mu)$. As μ increases, the mean of $a(n,\mu)$ remains unchanged, but $a(n,\mu)$ becomes less risky. Thus, by Jensen's inequality and the concavity of the power function $1/\sigma$, it follows that the second line is positive. Next, comparing the terms in the integrals on the third and fourth lines, we note that $\gamma_0/(n+\mu)+\tau^{1-\sigma}\gamma_1$ has the same mean but is riskier than $\gamma_0/(n+\mu)+\tau^{1-\sigma}a(n,\mu)$ for any given γ_0. Thus, Jensen's inequality and the convexity of the power function $-\rho$ imply that the fourth line exceeds (the absolute value of) the third line. Therefore, the derivative of $M(\hat{S}^d_{n+\mu})^{1/\sigma} - M(\hat{S}^c_{n+\mu})^{1/\sigma}$ with respect to μ is positive, which proves Lemma 2. $\square$

In the sector with common foreign shocks, the export cutoffs are given by (20). In the sector with idiosyncratic foreign shocks, $M(S^d_n) - M(S^d_{n-1})$ does not necessarily decrease with n and it is possible that $\varphi_n = \varphi_{n+1}$ for some n. Therefore, the export cutoffs satisfy

$$\varphi^d_n = \left\{ \frac{f_x}{k\left[M(S^d_n) - M(S^d_{n-1})\right]} \right\}^{1/(\sigma-1)}$$

if there are firms exporting to exactly n countries, and

$$\varphi^d_n < \left\{ \frac{f_x}{k\left[M(S^d_n) - M(S^d_{n-1})\right]} \right\}^{1/(\sigma-1)}$$

if no firm exports to exactly n countries. In view of Lemma 2 and given that $M(S^c_1) - M(S^c_0) = M(S^d_1) - M(S^d_0)$, this proves that $\varphi^d_1 \leq \varphi^c_1$ and $\varphi^d_n < \varphi^c_n$ for $n > 1$.

Part B: Proof that $\varphi^c_N < \varphi^g_1$.

For all $n > 0$

$$M(S^c_n) - M(S^c_{n-1}) > \left(1 - \tau^{1-\sigma}\right) M(S^c_0),$$

while

$$M(S^g_n) - M(S^g_{n-1}) > \left(1 - \tau^{1-\sigma}\right) M(S^g_0).$$

Since $M(S^c_0) = M(S^g_0)$, it follows that

$$M(S^c_N) - M(S^c_{N-1}) > M(S^g_1) - M(S^g_0),$$

which proves that $\varphi^c_N < \varphi^g_1$.

Together, part A and part B prove the proposition. ■