IZA DP No. 15127

Demographic Changes, Labor Supplies, Labor Complementarities, Calendar Annual Wages of Age Groups, and Cohort Life Wage Incomes

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ABSTRACT

Demographic Changes, Labor Supplies, Labor Complementarities, Calendar Annual Wages of Age Groups, and Cohort Life Wage Incomes

This paper analyzes the impact on age group wage differentials in a setting of imperfect labor substitution at different ages (years) of working life. We examine the wage prospect of assuming medium, high, and low levels of fertility during the population projection period (2020-2090). Main focus is on comparisons of selected Calendar year Age wage profiles and the comparisons of selected Cohort Lifetime wage profiles. The analytical results come from applying a CRESH Labor Aggregator to Age-group Labor supplies with a parametric calibration to register based micro data for Denmark. The results show Calendar year wage effects and Cohort wage effects from ageing that will not exist without non-zero Labor Complementarity elasticities, and are new contributions demonstrating the economic effects of large/small generations and cohort sizes. The impact of cohort size on the lifetime wage profile of its own cohort does depend on sizes of other cohorts, which are affected by the fertility rates underlying many cohorts. Hence, economic advantages of being a small cohort depend on fertilities and the sizes of many other existing cohorts.

JEL Classification: J1, O4, E2
Keywords: labor substitution, CRESH, demographic cohorts, lifetime wage incomes

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1 Introduction

Demographic changes (projections) affect the Population Age Distribution as well as size and Age Composition (absolutely and relatively) of the available Labor supplies from the relevant working age groups. This paper address economic implications of imperfect substitution and complementarity of the Labor services from different Labor Age groups.

The standard assumption in demographic macro modelling is that the aggregate labor variable is a simple sum of the homogeneous labor services of different age groups - which implies perfect substitution and same wage. This means that the influence (size effect) of aggregate labor supply by an increase in workers of a particular age-group is not affected by the Age distribution (relative number) of workers already in the labor force. For example, when younger workers are becoming relatively scarce, it makes sense to allow for their age-specific contribution to an Aggregate measure (Aggregator) of Labor supplies. Hence we allow for labor heterogeneity by specifying a parametric CRESH\(^1\) labor aggregator. This analytic Labor Aggregator function has implications for relative wages of both younger and older workers. In particular, if labor income is higher at younger ages and lower at older ages, then total Lifetime wage income of some generations (or cohorts) may be higher, while others are lowered. Various wage impacts are defined and calculated with lower/higher fertility of current and future calendar year generations.

In this paper, the purpose is to offer an analytic globally regular labor supply function (CRESH Labor Aggregator formed by any finite number of labor supply variables) with an empirical applications (parameter calibrations) to Danish micro data - and potential use for any country, where application of the principle of imperfect labor substitution is warranted. Our focus is next on investigating various micro and macro implications of projected demographic changes in this century (2010-2100) upon relative and absolute annual wages of 11 five-year Age groups of all working ages (15-69) in selected Calendar years and then give the lifetime labor incomes of some proper defined Cohorts.

Among the main new results with our analytically extended CRESH wage model formulations are the extensive CRESH demonstrations (scenarios) of comparative wages (relative/absolute) of all age groups for some calendar years \((t)\) in period (2020-2090),

\(^1\)CRESH stands for Constant Ratio of Elasticities of Substitution, Homothetic, (Hanoch, 1971).
as well as obtaining the Life time wage incomes for selected Cohorts (Generations) of different sizes, entering the Labor market in the year, T= 2010, 2015, 2020, 2030, 2035.

Design and estimations of Labor aggregator (supply) functions have a long history. Only a short literature review is given here. Dougherty (1972, p.1110-16) discuss Labor aggregation structures based on 8 non farm occupations or 8 educational (length) categories. The aggregation functional forms are single-level CES functions or many two-level CES aggregations.\(^2\) Leontief forms (fixed manpower requirements, \(\sigma = 0\)), Linear aggregation, \(\sigma = \infty\), were extreme (invalid) forms, and CD function (\(\sigma = 1\)) implied too little scope for substitution (inappropriate for aggregating labor). CES function were seen as improvements on these special forms of aggregation. Chiswick (1985, p.503) adhered to CES with moderately high elasticity (\(\sigma = 2.5\)) between each pair of factors (including labor (human capital) of at least two quality levels of salaried employees). For US, UK, Canada, Card and Lemieux (2001, p.709,725) estimated (\(\sigma\)) in the range: 4-6 (1/0.23, 1/0.17) for two CES subaggregates (High School, College) of workers from 7 age groups. Recently, Guest and Parr (2020, p.509) used, \(\sigma = 1\), \(\sigma = 2\), for Australian CES labor aggregates of 11 age groups. However, long ago Berndt and Christensen (1973, p.407) proved that a consistent CES aggregators at all points in factor space is equivalent to equality and constancy of all Allen-Uzawa partial elasticities of substitution (AUES, \(\sigma_{ij}\)). Evidently, the substitution elasticities between many labor services of different age-groups have never been the same or strictly constant (independent of labor supplies) anywhere.

Clearly, more sophisticated aggregator functions than CES are then to be considered. But functional complexity must be restrained to preserve sensible theoretical and empirical robust patterns of the substitution elasticities (\(\sigma_{ij}\)). It is here that the CRESH function of Hanoch (1971, p.697) enter as a proper aggregator of different (heterogeneous) labor services, since it allows relative substitution patterns of \(\sigma_{ij}\) between services to be preserved (remain constant). Moreover, with our focus on the consequences for the wage structure of sizeable changes in the age composition of the (exogenous, demographic) labor supplies, we need to see for CRESH functions also the (dual) partial complementarity elasticities (\(c_{ij}\)), Sato and Koizumi (1973, p.47), which link the relative and absolute

wage changes to variety in Labor supplies of Age-groups. These structure-analytic issues are probed jointly with CRESH calculations of $(\sigma_{ij})$ and $(c_{ij})$ in Appendices A-B.

Already Freeman (1979, p.301-303,313) estimated complementarity elasticities $(c_{ij})$ by the Trans-Log production function, using CPS (Current Population Survey) data tapes of individual (Micro) age-earnings (age-wage profiles). Our Micro (Personal Register) data on Danish labor supplies (annual full time equivalents) of age groups and annual wages are provided by Statistics Denmark (Department of Labor and Income).

The paper is organized as follows.

Section 2 presents in Table 1 and Figure 1 already known demographic trends in the Age composition of Populations in this century as the background for our economic analysis of relative wages and life time labor incomes. It describes Labor supplies from Microlevel (Register) data for year 2010 in Table 2, within a Macro framework - National Income Product Accounts (NIPA) for year 2010, shown in Table 3.

Section 3 presents the CRESH labor aggregator and the implications for age-wage profiles. It explains the methodology of calibrating the CRESH parameters to Labor Micro data of 2010; the calibrated CRESH model is validated on Micro data for 2013 in Table 4.

Section 4 calculates demographic - for Medium, Low, and High fertility from Table 1 - projected Labor Supplies of eleven Age groups, spanning working life of 55 years (15-69). It applies the CRESH model for the projected Labor supplies by showing the comparative wages (relative/absolute) of all age groups for calendar years $(t)$ in period (2020-2090). The main results are collected in Tables (5a-5c), Table 6, and exhibited in Fig. 2-6.

Section 5 demonstrates the CRESH calculations of Life time wage incomes of selected Cohorts, entering the Labor market in particular years $(T)$ of this century. The main results are explained and demonstrated by Table 7 and illustrated in Figures 7-12.

Further micro and macro aspects of Labor aggregation and CRESH Age-wage profiles are discussed in section 6 in reference to the literature on Wage structure from ‘Division of Labor’ by labor of various levels of experience, skills - Canonical Model, Appendix C.

Section 7 offer final comments/suggestions for teaming up Demography and Economics. Appendices (A,B,C) derive the basic CRESH Labor substitution elasticities and the new CRESH complementarity (Hicks-Sato) elasticities for all the Labor age-group wages.
2 Population age groups, Labor supplies and Wages

2.1 Demographic outlook, assumptions and future age groups

Let us briefly give the demographic outlook. Denmark, like most other developed countries, faces demographically further population ageing for some decades. Table 1 shows Danish Population age shares, \( n_i = N_i/N \), of three age groups: children (0-14), working age (15-69), and old age (70+) - under three Fertility ‘variants’ (Medium, Low, High), cf. Table 1a, as published by the United Nations Population Division (United Nations, 2015). For the three age shares \((n_i)\), males and females are combined. Life expectancy is the same under all fertility variants. The ‘Medium’ variant projections assume that the Total Fertility rate (TFR) slightly and monotonically increases from 1.730 to 1.876.

In the Medium variant scenario, the Working Population share \((n_{15-69})\) declines monotonically to a minimum (0.521) in year 2050, after which it monotonically recovers to (0.715), similar to its present size. The ‘Medium’ variant share numbers \((n_i)\) indicate an population ”ageing” or ”burden” problem for the next 20-40 years.

The Low variant numbers \((n_i)\) suggest in contrast that population ageing or ”burden” problems will occur after 2050. The High variant numbers \((n_i)\) show a remarkable stable population composition after 2025 - with even the old age (70+) share in balance.

When Fertility rate (TFR) is permanently less than 2.0, there would be long-run tendency for the total size of population \((N)\) to decline. However, if Life Expectancy is steadily increasing, then population \((N)\) may still increase, despite \((\text{TFR}) < 2.0\). Population \(N(t)\) for 2010-2100 in Medium Variant, Table 1, does not decline in any year.

The Danish Population \(N(2010) = 5.551\) million. For Medium Variant, projected numbers are: \(N(2020) = 5.776, N(2030) = 6.003, N(2050) = 6.299, N(2100) = 6.838\) million. Low Variant, Table 1, population eventually does decline. For Low Variant, projected numbers are: \(N(2020) = 5.732, N(2030) = 5.792, N(2050) = 5.603, N(2100) = 4.599\) million. High Variant, Table 1, population certainly does increase. For High Variant, projected numbers are: \(N(2020) = 5.819, N(2030) = 6.214, N(2050) = 7.154, N(2100) = 9.843\) million.

The three population age shares, \(n_i = N_i/N\), define a dependency rate of young/children (0-14) to working age population : \((d_y)\), and an old/age (70+) dependency rate to working
age population: \((d_o)\), and hence give the total dependency ratio: \((d)\), defined as:

\[
d_y = \frac{n_{0-14}}{n_{15-69}} = \frac{N_{0-14}}{N_{15-69}}, \quad d_o = \frac{n_{70+}}{n_{15-69}} = \frac{N_{70+}}{N_{15-69}}; \quad d = d_y + d_o; \quad \frac{1}{1 + d} = n_{15-69} \quad (1)
\]

which, as calculated in Table 1b, are exhibited in Fig. 1. Note that in Table 1b, e.g., Medium variant, 2010 : \(1/(1 + d) = (1/1.411) = 0.709 = n_{15-69}\), (age share), Table 1. Thus, total dependency ratio \((d)\) is uniquely related to \(n_{15-69}\), i.e., the columns \((d)\), Table 1b, tell essentially, for every variant, a similar story as \(n_{15-69}\) in Table 1, e.g. Medium variant, 2050 : small, 0.643 = \(n_{15-69}\), and high value of \(d = 0.555\); but monotonicity of \((d)\) in the Low variant seems more "dramatic" than the monotonicity of \(n_{15-69}\).

The projected dependency ratios, \(d_y, d_o\), (1), are dominated by the paths of \(n_{15-69}\), although \(n_{70+}\) exerts significant influence on \((d_o)\) in the Low variant. It is projections of the dependency ratio, \((d_o)\), that has attracted attention in the literature, Rojas (2005, p.466), Hu et al. (2000, p.117), Kitao (2015, p.38). When dependency ratio \((d_o)\) is seen redefined as: \(\tilde{d}_o = N_{65+}/N_{15-64}\) (retirement age, 65), projected sizes of these numbers \((\tilde{d}_o)\) appear in the literature more spectacular than \(d_o\) in Table 1b for Denmark: 2010-2100.

The Labour Force Participation rate, \(LFP\), is defined by, \(L_{15-69}/N_{15-69} = l_{15-69}\):

\[
LFP = \frac{L_{15-69}}{N_{15-69}} = l_{15-69} \quad (2)
\]

The Support ratio \((L/N)\), defined as the ratio of total Labor force (Labor supply) \((L)\) (Employment) to total Population \((N = N_{0-70+})\) is obtained from the dependency ratio \((d)\), cf. (1), and the Labour Force Participation \((LFP)\) rate, \(l_{15-69}\), (2), as follows:

\[
\frac{L}{N} = \frac{L_{15-69}}{N_{15-69}} = \frac{L_{15-69}}{N_{15-69}} \cdot \frac{N_{15-69}}{N} = l_{15-69} \cdot \frac{1}{1 + d} \quad (3)
\]

Hence with e.g., \(LFP\) for 2010 : \(l_{15-69} = 0.536\), we get by (3) for the Medium variant, the Support ratio in 2010 : \(L/N = 0.536 \cdot 0.709 = 0.38\). Evidently, for a given, \(l_{15-69}\), (constant LFP), the Support \((\text{Employment/Population})\) ratio \((L/N)\), (3) gives - for every fertility variant - the same scenario as the projected Working Population share: \(n_{15-69}\) in Table 1 - or inversely with the projected dependency ratio : \((d)\) in Table 1b.

The rising dependency ratio, \((d_o)\), implies that the Danish support ratio, \((L/N)\), falls from 0.38 (2010) to a level around 0.34 after 2050. The support ratios \((L/N)\) are now declining in many countries and are expected to continuously fall in the years until 2050.
Table 1a. Fertility and Life Expectancy Assumptions: Denmark, 2010-2100

<table>
<thead>
<tr>
<th></th>
<th>Medium variant</th>
<th>Low variant</th>
<th>High variant</th>
<th>Life Expectancy**</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Fertility*</td>
<td>Fertility*</td>
<td>Fertility*</td>
<td></td>
</tr>
<tr>
<td>2010-2015</td>
<td>1.730</td>
<td>1.730</td>
<td>1.730</td>
<td>8.507</td>
</tr>
<tr>
<td>2015-2020</td>
<td>1.761</td>
<td>1.511</td>
<td>2.011</td>
<td>8.779</td>
</tr>
<tr>
<td>2020-2025</td>
<td>1.785</td>
<td>1.385</td>
<td>2.185</td>
<td>9.053</td>
</tr>
<tr>
<td>2025-2030</td>
<td>1.804</td>
<td>1.304</td>
<td>2.304</td>
<td>9.344</td>
</tr>
<tr>
<td>2030-2035</td>
<td>1.817</td>
<td>1.317</td>
<td>2.317</td>
<td>9.653</td>
</tr>
<tr>
<td>2035-2040</td>
<td>1.829</td>
<td>1.329</td>
<td>2.329</td>
<td>9.987</td>
</tr>
<tr>
<td>2040-2045</td>
<td>1.841</td>
<td>1.341</td>
<td>2.341</td>
<td>10.304</td>
</tr>
<tr>
<td>2045-2050</td>
<td>1.848</td>
<td>1.348</td>
<td>2.348</td>
<td>10.609</td>
</tr>
<tr>
<td>2050-2055</td>
<td>1.854</td>
<td>1.354</td>
<td>2.354</td>
<td>10.903</td>
</tr>
<tr>
<td>2055-2060</td>
<td>1.858</td>
<td>1.358</td>
<td>2.358</td>
<td>11.205</td>
</tr>
<tr>
<td>2060-2065</td>
<td>1.862</td>
<td>1.362</td>
<td>2.362</td>
<td>11.505</td>
</tr>
<tr>
<td>2065-2070</td>
<td>1.865</td>
<td>1.365</td>
<td>2.365</td>
<td>11.798</td>
</tr>
<tr>
<td>2070-2075</td>
<td>1.868</td>
<td>1.368</td>
<td>2.368</td>
<td>12.091</td>
</tr>
<tr>
<td>2075-2080</td>
<td>1.870</td>
<td>1.370</td>
<td>2.370</td>
<td>12.396</td>
</tr>
<tr>
<td>2080-2085</td>
<td>1.872</td>
<td>1.372</td>
<td>2.372</td>
<td>12.715</td>
</tr>
<tr>
<td>2085-2090</td>
<td>1.874</td>
<td>1.374</td>
<td>2.374</td>
<td>13.028</td>
</tr>
<tr>
<td>2090-2095</td>
<td>1.875</td>
<td>1.375</td>
<td>2.375</td>
<td>13.353</td>
</tr>
<tr>
<td>2095-2100</td>
<td>1.876</td>
<td>1.376</td>
<td>2.376</td>
<td>13.691</td>
</tr>
</tbody>
</table>

Notes:
Variants differ only with respect to fertility assumptions.
* Fertility refers to number of children per woman.
** Life expectancy at age 80 for both sexes combined (number of years).

Table 1. Population Age Shares, \(n_i = N_i/N, \ i = 0-14, 15-69, 70+: \) Denmark, 2010-2100

<table>
<thead>
<tr>
<th>Medium Variant ((n_i))</th>
<th>Low Variant ((n_i))</th>
<th>High Variant ((n_i))</th>
</tr>
</thead>
<tbody>
<tr>
<td>0-14</td>
<td>15-69</td>
<td>70+</td>
</tr>
<tr>
<td>15-69</td>
<td>70+</td>
<td></td>
</tr>
<tr>
<td>0.180</td>
<td>0.709</td>
<td>0.111</td>
</tr>
<tr>
<td>0.173</td>
<td>0.702</td>
<td>0.125</td>
</tr>
<tr>
<td>0.163</td>
<td>0.689</td>
<td>0.148</td>
</tr>
<tr>
<td>0.160</td>
<td>0.683</td>
<td>0.157</td>
</tr>
<tr>
<td>0.165</td>
<td>0.669</td>
<td>0.166</td>
</tr>
<tr>
<td>0.168</td>
<td>0.656</td>
<td>0.176</td>
</tr>
<tr>
<td>0.167</td>
<td>0.644</td>
<td>0.189</td>
</tr>
<tr>
<td>0.164</td>
<td>0.642</td>
<td>0.194</td>
</tr>
<tr>
<td>0.161</td>
<td>0.643</td>
<td>0.196</td>
</tr>
<tr>
<td>0.160</td>
<td>0.639</td>
<td>0.201</td>
</tr>
<tr>
<td>0.161</td>
<td>0.637</td>
<td>0.202</td>
</tr>
<tr>
<td>0.163</td>
<td>0.636</td>
<td>0.201</td>
</tr>
<tr>
<td>0.162</td>
<td>0.629</td>
<td>0.209</td>
</tr>
<tr>
<td>0.160</td>
<td>0.625</td>
<td>0.215</td>
</tr>
<tr>
<td>0.157</td>
<td>0.621</td>
<td>0.222</td>
</tr>
<tr>
<td>0.155</td>
<td>0.623</td>
<td>0.222</td>
</tr>
<tr>
<td>0.155</td>
<td>0.621</td>
<td>0.224</td>
</tr>
<tr>
<td>0.155</td>
<td>0.617</td>
<td>0.228</td>
</tr>
<tr>
<td>2010</td>
<td>2015</td>
<td>2020</td>
</tr>
</tbody>
</table>

Total population (both sexes combined) by five-year age group.

The graphics of the Danish dependency ratio \((d)\) in Table 1b is exhibited in Figure 1 for the three UN demographic\(^3\) variants (Medium, High, Low).

\(^3\)The declining fertility in recent decades and hence the falling dependency ratio, \((d_y)\), have in several countries dominated the rising \((d_d)\), such that Support ratios \((L/N)\), \((3)\), in some countries have in certain periods until 2010 actually increased - and been called "demographic (fertility) dividends.”

We shall in Table 6 see a few economic illustrations of this “dividend” in the Danish Medium fertility variant for some years after the ‘minima’ of the ‘Working Age Population’ share, \(n_{15-69}\), in 2050.

Some illustrations of US dependency ratios and Support ratios are seen in, Cutler et al. (1990, p.5,8).
Fig. 1. Danish Dependency ratio - d - for Medium, Low, High fertility, 2010-2100.

Source: Total dependency ratio (d), (1), with the numbers from Table 1b.

<table>
<thead>
<tr>
<th>Year</th>
<th>d_m</th>
<th>d_l</th>
<th>d_h</th>
</tr>
</thead>
<tbody>
<tr>
<td>2010</td>
<td>0.254</td>
<td>0.157</td>
<td>0.411</td>
</tr>
<tr>
<td>2015</td>
<td>0.246</td>
<td>0.178</td>
<td>0.424</td>
</tr>
<tr>
<td>2020</td>
<td>0.237</td>
<td>0.215</td>
<td>0.452</td>
</tr>
<tr>
<td>2025</td>
<td>0.234</td>
<td>0.230</td>
<td>0.464</td>
</tr>
<tr>
<td>2030</td>
<td>0.247</td>
<td>0.248</td>
<td>0.495</td>
</tr>
<tr>
<td>2035</td>
<td>0.256</td>
<td>0.268</td>
<td>0.524</td>
</tr>
<tr>
<td>2040</td>
<td>0.259</td>
<td>0.293</td>
<td>0.552</td>
</tr>
<tr>
<td>2045</td>
<td>0.255</td>
<td>0.302</td>
<td>0.557</td>
</tr>
<tr>
<td>2050</td>
<td>0.250</td>
<td>0.305</td>
<td>0.555</td>
</tr>
<tr>
<td>2055</td>
<td>0.250</td>
<td>0.315</td>
<td>0.565</td>
</tr>
<tr>
<td>2060</td>
<td>0.253</td>
<td>0.317</td>
<td>0.570</td>
</tr>
<tr>
<td>2065</td>
<td>0.256</td>
<td>0.316</td>
<td>0.572</td>
</tr>
<tr>
<td>2070</td>
<td>0.258</td>
<td>0.332</td>
<td>0.590</td>
</tr>
<tr>
<td>2075</td>
<td>0.256</td>
<td>0.344</td>
<td>0.600</td>
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<tr>
<td>2080</td>
<td>0.253</td>
<td>0.357</td>
<td>0.610</td>
</tr>
<tr>
<td>2085</td>
<td>0.249</td>
<td>0.356</td>
<td>0.605</td>
</tr>
<tr>
<td>2090</td>
<td>0.250</td>
<td>0.361</td>
<td>0.611</td>
</tr>
<tr>
<td>2095</td>
<td>0.251</td>
<td>0.370</td>
<td>0.621</td>
</tr>
<tr>
<td>2100</td>
<td>0.254</td>
<td>0.383</td>
<td>0.637</td>
</tr>
</tbody>
</table>

Source: The dependency ratios, d_y, d_o, d, are defined in (1) and calculated by Table 1.

The overall dependency ratio, d, (1), is rising for each fertility scenario in Figure 1. But in the high fertility scenario, the dependency ratio (d) is 'stationary' in the 50 years from 2040 to 2090. In the low fertility scenario, we find a significant increase in the dependency ratio (d) from 2060 onwards, as the delayed impact of prior low fertilities.
As well-known, a *Fertility* rate of 2.1 (children per woman) on average is usually necessary for reproduction of population levels; increasing Life Expectancy modifies the requirement. As mentioned above Total Danish Population size, $N(t)$, never declines, but slowly increases during projection period 2020 – 2090 under *Medium* Variant of Tables (1a, 1), cf. Table 6 (Row 4) below - that also shows that $N(t)$ declines after 2040 in the *Low* Variant, and clearly $N(t)$ increases (nearly doubles by 2090) for the *High* Variant.

Having presented the United Nations projections of the evolution of the Danish demographic structure 2010-2100 in Tables (1,1b,6), Fig. 1, with the general concepts and terminology, (1-3) - similar descriptions apply to any UN country - we restate (for later use) overall LFP, (2), in terms of 11 age-specific labor participation rates, $l_i = \frac{L_i}{N_i}$:

$$LFP = \frac{L_{15-69}}{N_{15-69}} = l_{15-69} = \sum_{i=1}^{11} \frac{L_i}{N_i} \cdot \frac{N_i}{N_{15-69}} = \sum_{i=1}^{11} l_i \cdot \bar{n}_i ; \; L_i(t) = l_i \cdot N_i(t), \; i = 1, , 11 \; (4)$$

With proper chosen $l_i$ as exogenous parameters, we can derive age-specific Labor supplies in any calendar year (t), $L_i(t)$, (4), from the evolutions of, $N_i(t) = n_i(t) \cdot N(t)$, and $N(t)$.

Introducing imperfect labor substitution/complementarity between age-specific Labor group supplies $L_i(t)$, (4), significantly changes the relative wages within the Total Labor force (supply), $L_{15-69}(t)$, over time. We will use a suitable Labor economic model analytically designed to generate/explain such Age Group Wage Differentials. Section 3 presents a CRESH model with distinct parameters for labor age group supplies.

---

4The assumption of perfect substitution of labor among age groups has been challenged, tested empirically and relaxed in a variety of modelling approaches, Prskawetz et al. (2008), Guest (2007), Creedy and Guest (2007), Guest and Shacklock (2005), Hamermesh (1993), Lam (1989).
2.2 Labor supplies of age groups, Micro wage data, and NIPA

To apply the CRESH labor *aggregator* for empirically analyzing age-wage profiles, the CRESH parameters must be properly related to specific labor supplies and wage data. Here we use Micro and Macro data for Denmark in the year 2010 - to be explained and shown in Tables 2-3. Similar Micro data 2013 are used for CRESH aggregator validation.

As provided from the United Nations World Population Prospects, 2015 Revision, total Population (both sexes) by five year age groups (United Nations, 2015), the Danish Population numbers, $N_i$ (column 2, Table 2) are: The demographic sizes of our eleven 5-year working age groups (15-69), and the young (0-14) and old age (70+) groups, i.e., the absolute sizes of the age groups in 2010, corresponding to age shares ($n_i$) in Table 1.

Within the eleven 5-year working age groups, the oldest $N_i$, (65-69), soon fully retired, are born in (1941-1945). In (1945,1946), the number of births peaked with (95-96.000). The post-war (1946-1950), generation are seen in $N_i$ (60-64). Birth rates started to slowly decline in the 1950’s ; the Danish economy was stagnating until 1957, and net emigration occurred, as can be seen from the $N_i$ (55-59) numbers, which also partly reflect a negative ‘echo’ of the smaller depression year generations of 1930’s. In contrast, a positive ‘echo’ of two war-postwar generations above and prosperous full-employment years of 1960’s are reflected in sizes $N_i$ (45-49), $N_i$ (40-44) of the two generations, (1961-1965), (1966-1970). The European oil-shock recession and unemployment years, (1976-1980), (1981-1985), are reflected in the small Danish numbers of the, $N_i$ (30-34), $N_i$ (25-29).

From the beginning of 1990’s, revenues from North Sea oil - as in UK and Norway - contributed to remove deficits of Danish international and public sector accounts. Child benefits were subsequently increased; significant immigration also began to matter in these years. They are explanatory population elements of a turn-around, seen in sizes of both $N_i$ (20-24), and the youngest age group, $N_i$ (15-19), born in (1991-1995).

We must next explain the Labor supplies used and their associated wages in 2010. The Labor age group numbers (Labor years) $L_i$ (column 3, Table 2) are Danish full time workers (equivalents, 1924 hours) - with age distribution ($\lambda_i$), (col. 4), and their average annual wages ($w_i$), (column 6). These Microlevel data (personal register) were provided to the authors by Statistics Denmark. These Register data, however, were excluding
agriculture, fishery, and all firms with less than 10 full-time employees.

We calibrate our model to *National Accounting* data for Denmark in 2010, which implies that the sum of \( L_i \) (col. 5) must equal aggregate employment (Table 3, row 1) of 2112472 full time workers (*Labor years*). We use the age-specific labour fractions \( \lambda_i \) of *Register* data (column 4) to gross up the values of \( L_i \) such that the total of \( L_i \) (col. 5) is equal to: \( L = 2112472 \). The age-specific wages \( w_i \) in Table 2 (col. 6) of *Register* data are multiplied by the adjusted (Total) Labor numbers, \( L_i \), (col.5), and summed to give the aggregate Wage Bill, which is 924.3 Billion DKK (col. 8). The aggregate Annual wage, \( w \), per unit of \( L \) is then found by dividing the aggregate Wage Bill (col. 8) by \( L \), which gives: \( w (2010) = 437552 \) DKK \( \equiv W_A \); cf. (41), Tables (5a, 5b).

<table>
<thead>
<tr>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Age (i)</td>
<td>( N_i )</td>
<td>( L_i )</td>
<td>( \lambda_i )</td>
<td>( L_i )</td>
<td>( w_i )</td>
<td>( w_i/\lambda_i )</td>
<td>( w_iL_i )</td>
<td>( N_i )</td>
<td>( L_i/N_i )</td>
</tr>
<tr>
<td>15-19</td>
<td>353109</td>
<td>21138</td>
<td>0.0143</td>
<td>30199</td>
<td>187005</td>
<td>0.4554</td>
<td>5.647</td>
<td>0.0061</td>
<td>0.0855</td>
</tr>
<tr>
<td>20-24</td>
<td>331419</td>
<td>77504</td>
<td>0.0524</td>
<td>110731</td>
<td>273220</td>
<td>0.6653</td>
<td>30.254</td>
<td>0.0327</td>
<td>0.3341</td>
</tr>
<tr>
<td>25-29</td>
<td>310515</td>
<td>117177</td>
<td>0.0792</td>
<td>167411</td>
<td>358262</td>
<td>0.8724</td>
<td>59.977</td>
<td>0.0649</td>
<td>0.5391</td>
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<tr>
<td>30-34</td>
<td>347261</td>
<td>170826</td>
<td>0.1155</td>
<td>244059</td>
<td>410668</td>
<td>1.0000</td>
<td>100.227</td>
<td>0.1084</td>
<td>0.7028</td>
</tr>
<tr>
<td>35-39</td>
<td>388101</td>
<td>202853</td>
<td>0.1372</td>
<td>289186</td>
<td>449679</td>
<td>1.0950</td>
<td>130.324</td>
<td>0.1410</td>
<td>0.7468</td>
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<tr>
<td>40-44</td>
<td>408902</td>
<td>214618</td>
<td>0.1451</td>
<td>306624</td>
<td>471118</td>
<td>1.1472</td>
<td>144.456</td>
<td>0.1563</td>
<td>0.7499</td>
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<td>45-49</td>
<td>405079</td>
<td>211983</td>
<td>0.1434</td>
<td>302860</td>
<td>472491</td>
<td>1.1505</td>
<td>143.099</td>
<td>0.1548</td>
<td>0.7477</td>
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<tr>
<td>50-54</td>
<td>366102</td>
<td>188472</td>
<td>0.1275</td>
<td>269270</td>
<td>471381</td>
<td>1.1478</td>
<td>126.929</td>
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<td>0.7355</td>
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<tr>
<td>55-59</td>
<td>350020</td>
<td>169527</td>
<td>0.1147</td>
<td>242203</td>
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<td>1.1243</td>
<td>111.832</td>
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<td>60-64</td>
<td>368451</td>
<td>88874</td>
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<td>126974</td>
<td>478708</td>
<td>1.1657</td>
<td>60.783</td>
<td>0.0658</td>
<td>0.3446</td>
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<td>65-69</td>
<td>309369</td>
<td>15626</td>
<td>0.0106</td>
<td>22325</td>
<td>483248</td>
<td>1.1767</td>
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<td>2112472</td>
<td>437552</td>
<td>924.317</td>
<td>1.0000</td>
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</tr>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>70+</td>
<td>615547</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>5550959</td>
<td>2112472</td>
<td>166515</td>
<td>924.317</td>
<td>0.3806</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Source: UNITED NATIONS (UN), see Table 1; STATISTICS DENMARK (Department of Labor and Income), Copenhagen.

- Column 1: Age groups, \( i = 1, \ldots, 16 \); \( i = 1, 15-19, \ldots, 11: 65-69 \); \( i = 12: 0-14, i = 13: 70+ \).
- Column 2: Population (total) in age groups; UN Population Data.
- Column 3: Full time workers (Annual equivalents, 1924 hours), Labor services in Labor years; Microlevel (personal register) data.
- Column 4: Labor age group distribution - Fractions, \( \lambda_i \), same in column 3 and column 5.
- Column 5: Total full time workers in labor age groups, \( L_i = \lambda_iL \) (\( L = 2112472 \) Total full time workers); stat.bank, DB07, ERHV1.
- Column 6: Average annual wages of labor age groups \( w_i \) in column 3 and 5; \( w = 437552 \) Billion DKK / 2112472.
- Column 7: Relative annual wages, age group wage profile - generated by the Microlevel (personal register) data in column 6.
- Column 8: Total wage incomes of age group, (\( \lambda_i \)) Billion DKK; stat.bank, DB07, ERHV1, Total wage sum = 930.286 Billion DKK.
- Column 9: Age group wage income shares \( \lambda_i \) (shares in the total wage bill, 924.317 Billion DKK).
- Column 10: Labor participation rates (LPR) of age groups - derived from column 2 and column 5.
Table 3. National Income Accounts - Data for Denmark, 2010

<table>
<thead>
<tr>
<th>Descriptions</th>
<th>Symbols</th>
<th>Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total (equivalent) full time workers</td>
<td>( L )</td>
<td>2112472 Labor years</td>
</tr>
<tr>
<td>Average (aggregate) wage per labor year (man-year)</td>
<td>( w )</td>
<td>437552 DKK</td>
</tr>
<tr>
<td>Total wage incomes</td>
<td>( wL )</td>
<td>924.3 Billion DKK</td>
</tr>
<tr>
<td>Net capital (rental) incomes</td>
<td>( rK )</td>
<td>363.8 Billion DKK</td>
</tr>
<tr>
<td>Net Factor Incomes (NFI) – Net Domestic Value Added</td>
<td>( Y = wL + rK )</td>
<td>1228.1 Billion DKK</td>
</tr>
<tr>
<td>Capital consumption/depreciation</td>
<td>( \delta K )</td>
<td>318.1 Billion DKK</td>
</tr>
<tr>
<td>Gross Factor Incomes (GFI) – Gross Domestic Value Added</td>
<td>( GFI )</td>
<td>15462 Billion DKK</td>
</tr>
<tr>
<td>Net capital stock</td>
<td>( K )</td>
<td>57415 Billion DKK</td>
</tr>
<tr>
<td>Net capital-output ratio</td>
<td>( \nu = K/Y )</td>
<td>4.67</td>
</tr>
<tr>
<td>Net capital-labour ratio</td>
<td>( k = K/L )</td>
<td>2.72 Billion DKK</td>
</tr>
<tr>
<td>Average labour productivity</td>
<td>( y = Y/L )</td>
<td>581972 DKK</td>
</tr>
<tr>
<td>Depreciation rate</td>
<td>( \delta = \delta K/K )</td>
<td>0.055 percent</td>
</tr>
<tr>
<td>Net real interest rate</td>
<td>( r = rK/K )</td>
<td>0.053 percent</td>
</tr>
<tr>
<td>Gross real interest rate</td>
<td>( r + \delta )</td>
<td>0.108 percent</td>
</tr>
<tr>
<td>Wage share of net factor income, ( w(Y/L) )</td>
<td>( \varepsilon_w = wY/L )</td>
<td>0.752</td>
</tr>
<tr>
<td>Capital share of net factor income, ( r(Y/K) )</td>
<td>( \varepsilon_r = rY/K )</td>
<td>0.248</td>
</tr>
<tr>
<td>Factor compensation, Asset income (net), to rest of world</td>
<td>( O )</td>
<td>29.6 Billion DKK</td>
</tr>
<tr>
<td>Net National Income, in factor prices</td>
<td>( Y = O + NNI )</td>
<td>1257.7 Billion DKK</td>
</tr>
<tr>
<td>Gross National Income, in factor prices</td>
<td>( C + Tr + S = GNI )</td>
<td>1575.8 Billion DKK</td>
</tr>
<tr>
<td>Consumption (private + public), in factor prices</td>
<td>( C )</td>
<td>1104.9 Billion DKK</td>
</tr>
<tr>
<td>Transfers to rest of world, net</td>
<td>( Tr )</td>
<td>36.6 Billion DKK</td>
</tr>
<tr>
<td>Gross National Saving</td>
<td>( S )</td>
<td>434.3 Billion DKK</td>
</tr>
<tr>
<td>Gross Domestic Investment, in factor prices</td>
<td>( I )</td>
<td>331.3 Billion DKK</td>
</tr>
<tr>
<td>Balance of payment, current account, Asset accumulation</td>
<td>( S - I = BP )</td>
<td>103.0 Billion DKK</td>
</tr>
<tr>
<td>Consumption ratio, NNI</td>
<td>( C/NNI )</td>
<td>0.879</td>
</tr>
<tr>
<td>Consumption ratio, NFI</td>
<td>( C/NFI )</td>
<td>0.900</td>
</tr>
<tr>
<td>Consumption per capita, in factor prices</td>
<td>( C/N )</td>
<td>199350 DKK</td>
</tr>
<tr>
<td>Net National Income per capita, in factor prices</td>
<td>( NNI/N )</td>
<td>226573 DKK</td>
</tr>
<tr>
<td>Annual wage income per capita, in factor prices</td>
<td>( wL/N )</td>
<td>166515 DKK</td>
</tr>
<tr>
<td>Support Ratio</td>
<td>( L/N )</td>
<td>3806</td>
</tr>
<tr>
<td>Net Factor Income (NFI) per capita, ( Y/NI = (Y/L)(L/N) )</td>
<td>( wL/C )</td>
<td>221499 DKK</td>
</tr>
<tr>
<td>Annual wage income-consumption ratio</td>
<td>( wL/C )</td>
<td>0.837</td>
</tr>
<tr>
<td>Gross national saving rate</td>
<td>( S/GNI )</td>
<td>0.276</td>
</tr>
<tr>
<td>Net national saving rate</td>
<td>( (S - \delta K/NI) )</td>
<td>0.092</td>
</tr>
<tr>
<td>Gross domestic investment rate</td>
<td>( I/GFI )</td>
<td>0.214</td>
</tr>
</tbody>
</table>


Labour Force Participation (LFP) rate (4), Support ratio (3) - cf. Table 2, col.10,1 - are:

\[
LFP = \frac{L_{15-69}}{N_{15-69}} = l_{15-69} = 0.5364; \quad \frac{L}{N} = \sum_{i=1}^{11} l_i \cdot n_i = \frac{11}{11} \frac{L_i}{N_i} = l_{15-69} \cdot n_{15-69} \quad (5)
\]

\[
\frac{L}{N} = l_{15-69} \cdot n_{15-69} = 0.5364 \cdot 0.7095 = 0.3806; \quad \frac{Y}{N} = \frac{Y}{L} \cdot \frac{L}{N}; \quad \frac{C}{N} = \frac{C}{Y} \cdot \frac{Y}{N} \quad (6)
\]

Support ratio \( L/N \) (5-6) is \( n_i \)-weighted age-specific, \( l_i \). Support ratio: a multiple of \( LFP \).

The per capita sizes of National Income, Consumption ratios, \( Y/N, C/N \), and their decomposition in (6) are seen in NIPA, Table 3 (row 31,27) - summarized in Table 3a.
Thus Micro based employment - full time equivalents - and wage data in Table 2 (col. 5,6,8), correspond exactly to Macro (National Income) data in Table 3 (Row 1-3, 29-31), for calendar year, 2010.\(^5\) Short version of Table 3 is seen in Table 3a - template to Table 6 - as calendar year summary of labor model results in Tables (5a, 5b).

<table>
<thead>
<tr>
<th>Table 3a. Population Age Groups, Labor Supply, Support Ratios, Incomes per capita: Denmark</th>
</tr>
</thead>
<tbody>
<tr>
<td>2010</td>
</tr>
<tr>
<td>2.</td>
</tr>
<tr>
<td>3.</td>
</tr>
<tr>
<td>4.</td>
</tr>
<tr>
<td>5.</td>
</tr>
<tr>
<td>6.</td>
</tr>
<tr>
<td>7.</td>
</tr>
<tr>
<td>8.</td>
</tr>
<tr>
<td>9.</td>
</tr>
<tr>
<td>10.</td>
</tr>
</tbody>
</table>

Table 3a/Table 6, Rows 1-4 show - in absolute quantitative form (Total numbers) for calendar years - the consequences of the demographic changes described in Tables (1,1a,1b). Rows 5-7 show, respectively, the absolute sizes of the total labor force, \(L(t)\), (supply), the sizes of LFP \((t)\) rate, (5), and the sizes of Support Ratio, \(L(t)/N(t)\), (6). Row 8 shows the Total Wage Income of \(L(t)\), [All age groups, in calendar year \((t)\)] working in any year \((t)\) with 11 age-specific participation rates : \(l_i(t) = l_i(2010)\), Table 2 (col. 10). Row 9 shows the Total Wage Income per capita, \(N(t)\), which - with the macroeconomic structure, technology levels, productivity conditions, \((Y/L), (K/L)\), of Table 3, (2010) - is equivalent to : \(L(t)/N(t) \cdot w(2010) = \text{Support ratio} \cdot W_A\).\(^6\) Row 10 shows National Income per capita, \(Y(t)/N(t)\), which is here a simple proportionality of Row 9, cf. (6).

The labor market equilibrium model with CRESH Aggregator functions - underlying all results in Tables 4-7 - must now be established and justified, theoretically and empirically, in Section 3 and Appendix A: Labor Substitution and Complementarity.

\(^5\)The year 2010 - as benchmark for our projected population variants, annual labor supply and wage income comparisons - is chosen for various reasons. It takes several years before the final revision of National Income Accounting (NIA) is completed. The Financial Crisis years (2008-2009) were unsuitable as benchmark years. For year 2010 the final revision came out in 2015. The processing of the Micro register databases for corresponding employment and wage data for 2010 began after NIA revision 2015. We cannot wait for getting reliable revised NIA for 2015 or later - as our model benchmark year.

Remark. The overall sizes of LFP = \(l_{15-69}\) in (5), Tables (2,4,6) look small ; the age-specific LFP, \(l_i\), Table 2 show that \(l_1, l_{11}\) give small \(l_{15-69}\) (Age group 65-69, \(l_{11}\), is seldom included in reported LFP).

\(^6\)As will be explained by Factor (Labor) cost (income) functions and duality theory in Appendix A.
3  CRESH Labor supply, Relative wages, Annual wage

Hanoch (1971, p. 697) introduced a globally regular CRESH implicit production (aggregator) function, \( F(Y, X_1, X_2, \ldots, X_M) = 0 \). Our CRESH function, \( F(L_A, L_1, L_2, \ldots, L_M) \), as seen in equation (7), is homogeneous of degree zero - and \( F \) determines implicitly the Labor Aggregate variable, \( L_A \), (Total Labor Supply), from the distinct (heterogeneous) Labor services, \( (L_1, L_2, \ldots, L_M) \), (M Labor Supplies), as stated in the expression:

\[
F(L_A, L_1, L_2, \ldots, L_M) = \gamma \sum_{i=1}^{M} \alpha_i \left[ \frac{L_i}{L_A} \right]^{\rho_i} - 1 = 0 \quad (7)
\]

with

\[
\gamma > 0; \quad \forall i : \alpha_i > 0, \quad \sum_{i=1}^{M} \alpha_i = 1; \quad \forall i : 0 < \rho_i \leq 1 \quad \text{or} \quad \rho_i < 0 \quad (8)
\]

Labor services (flows), \( (L_1, L_2, \ldots, L_M) \), may be measured in hours, working-weeks, or labor years. As in Table 2-3, we use as Labor unit: Labor years; the total flow variable (\( L_A \)) is also measured in Labor years. Thus ratios, \( \left( \frac{L_i}{L_A} \right) \), in (7) are unit-free (pure numbers), implying, too, that all parameters in (8) are unit-free (pure numbers).

For \( \forall i: \rho_i = \rho \), we get CES functions by (7-8), and CD as the limit function (\( \rho = 0 \)),

\[
\rho_i = \rho : L_A = \gamma^{\frac{1}{\rho}} \left[ \sum_{i=1}^{M} \alpha_i L_i^\rho \right] \gamma^{\frac{1}{\rho}} = \frac{1}{\rho} \sum_{i=1}^{M} \alpha_i L_i \quad ; \quad \rho = 1, L_A = \gamma \sum_{i=1}^{M} \alpha_i L_i \quad ; \quad \rho = 0, L_A = \gamma \prod_{i=1}^{M} L_i^{\alpha_i} \quad (9)
\]

Parameter restrictions (8) ensure that CRESH equation (7) represents a unique implicit Labor Aggregator function, \( L_A = f(L_1, L_2, \ldots, L_M) \), that is homogeneous of degree one, and is globally regular, i.e. for all \( L_i > 0 \), \( f(\ldots) \) is positive, non-decreasing, concave, with a negative semi-definite Hessian matrix, \( \frac{\partial^2 f}{\partial L_i \partial L_j} \).

\[
\forall L_i > 0 : L_A = f(L_1, L_2, \ldots, L_M) > 0 ; \quad \frac{\partial f}{\partial L_i} > 0 , \quad \frac{\partial^2 f}{\partial L_i \partial L_j} < 0 ; \quad L_A = \sum_{i=1}^{M} \frac{\partial f}{\partial L_i} L_i \quad (10)
\]

The CRESH function, \( F(L_A, L_1, L_2, \ldots, L_M) \), in (7) has the first-order derivatives,

\[
\frac{\partial F}{\partial L_i} = \frac{\gamma \alpha_i \rho_i (L_i/L_A)^{\rho_i-1}}{L_A}, \quad i = 1, \ldots, M ; \quad \frac{\partial F}{\partial L_A} = - \frac{\gamma \sum_{i=1}^{M} \alpha_i \rho_i (L_i/L_A)^{\rho_i}}{L_A} \quad (11)
\]

Marginal contributions of \( L_i \) to \( L_A \) : \( \frac{\partial L_A}{\partial L_i} = \frac{\partial f}{\partial L_i} \), and marginal rates of substitution (MRS) are given by implicit differentiation of \( F(L_A, L_1, L_2, \ldots, L_M) \), i.e. we get by (10-11):

\[
\frac{\partial L_A}{\partial L_i} = \frac{\partial f}{\partial L_i} = - \frac{\partial F/\partial L_i}{\partial F/\partial L_A} = \frac{\alpha_i \rho_i (L_i/L_A)^{\rho_i-1}}{\sum_{i=1}^{M} \alpha_i \rho_i (L_i/L_A)^{\rho_i}} > 0 , \quad i = 1, \ldots, M \quad (12)
\]
\[
\frac{dL_i}{dL_j} = MRS = \frac{\partial f/\partial L_i}{\partial f/\partial L_j} = \frac{\alpha_i\rho_i}{\alpha_j\rho_j} \frac{(L_i/L_A)^{\rho_i-1}}{(L_j/L_A)^{\rho_j-1}}, \quad i \neq j
\]  
\hspace{1cm} (13)

The elasticities, \( E(L_A, L_i), \) shares \((\varepsilon_i), \) add up to 1 by the degree of homogeneity in (10),

\[
\varepsilon_i = E(L_A, L_i) = \frac{\partial L_A}{\partial L_i} L_A = \frac{\alpha_i\rho_i}{\sum_{i=1}^{M} \alpha_i\rho_i} \frac{(L_i/L_A)^{\rho_i}}{1}; \quad \sum_{i=1}^{M} \varepsilon_i = 1 \hspace{1cm} (14)
\]

**Relative wages** - relative factor prices, (13) - must reflect their MRS. Hence CRESH relative wages, \((\frac{w_i}{w_j}),\) CRESH relative wage income shares, \((\frac{\varepsilon_i}{\varepsilon_j} = \frac{w_i}{w_j}\frac{L_i}{L_j}),\) become by (13-14):

\[
\frac{w_i}{w_j} = \frac{\alpha_i\rho_i}{\alpha_j\rho_j} \frac{(L_i/L_A)^{\rho_i-1}}{(L_j/L_A)^{\rho_j-1}} = \frac{\alpha_i\rho_i}{\alpha_j\rho_j} L_i^{\rho_i-1} L_A^{\rho_j-\rho_i}, \quad i \neq j; \quad \frac{\varepsilon_i}{\varepsilon_j} = \frac{\alpha_i\rho_i}{\alpha_j\rho_j} \frac{(L_i/L_A)^{\rho_i}}{(L_j/L_A)^{\rho_j}} \hspace{1cm} (15)
\]

These CRESH expressions emphasize the **relative wage effects** of particular labor supplies (pair), \(L_i, L_j,\) the substitution parameters, \(\rho_i, \rho_j,\) and the relative intensity parameters, \(\alpha_i, \alpha_j.\) Via **Total** variable \(L_A,\) all variables \(L_i,\) and all parameters in (7) affect \((\frac{w_i}{w_j}), (15).\)

The special cases of (15) for the CD-CES family (9) become \((1 - \rho = \frac{1}{\sigma}):\)

\[
CD: \hspace{0.5cm} \frac{w_i}{w_j} = \frac{\alpha_i}{\alpha_j} \frac{L_j}{L_i}; \quad CES: \hspace{0.5cm} \frac{w_i}{w_j} = \frac{\alpha_i}{\alpha_j} \left[ \frac{L_i}{L_j} \right]^{\sigma-1}; \quad Linear: \hspace{0.5cm} \frac{w_i}{w_j} = \frac{\alpha_i}{\alpha_j}; \quad i \neq j \hspace{1cm} (16)
\]

If (9) takes the linear form, **relative wages** (16) depend only on relative intensity parameters, \(\alpha_i, \alpha_j,\) whereas, \(L_i, L_j,\) also affect CD, CES, (16). On CRESH aggregator, see Conlon (1993); for discussions of empirical estimation of CRESH, see Weiss (1977, p.765).

**Changes in relative wages, \(w_i/w_j, (15),\) are smaller, the higher is the value of \(\rho_i.\)** Intuitively, the more flexible a labor supply is (higher value of \(\rho_i),\) the smaller change in its relative wage is required to clearing the labor markets (supply-demand equilibrium) for the given change in supply of the labor service, \(L_i.\)

The CRESH elasticity of the wage ratio, \((w_i/w_j),\) with respect to the labor supply \((L_i),\) is simply obtained from, (15), (14), (by elasticity rules for composite functions):

\[
E\left[\frac{w_i}{w_j}, L_i\right] = \rho_i - 1 + (\rho_j - \rho_i) \varepsilon_i < 0; \quad E\left[\frac{w_j}{w_i}, L_i\right] = 1 - \rho_i + (\rho_i - \rho_j) \varepsilon_i > 0 \hspace{1cm} (17)
\]

where \(\varepsilon_i\) is labor income share of \(L_i.\) Thus by (17), increasing \(L_i\) will always decrease the CRESH relative wage of \(L_i;\) but the higher \(\rho_i\) is, the smaller is the percentage decline in \((w_i/w_j);\) a higher \(\rho_j\) has a similar effect on diminishing the decline in \((w_i/w_j)\) as \(\rho_i.\) Moreover, a larger \(L_i\) will always increase the CRESH relative wages of \(L_j\) (other labor groups compared to \(L_i);\) the higher \(\rho_i\) is, the larger is the relative increase in \((w_j/w_i);\)
the effect of higher \(\rho_j\) gives a smaller increase in \((w_j/w_i);\) as a result of larger \(L_i.\)
By (15) - and using the same elasticity rules above - we also here note that,

$$ E \left[ \frac{\varepsilon_i}{\varepsilon_j}, L_i \right] = \rho_i + (\rho_j - \rho_i) \varepsilon_i > 0 ; \quad E \left[ \frac{\varepsilon_j}{\varepsilon_i}, L_i \right] = -\rho_i + (\rho_i - \rho_j) \varepsilon_i < 0 \quad (18) $$

Thus, in contrast to their relative wages in (17), the relative labor shares of $L_i$ in (18), always increases with larger $L_i$; moreover, the CRESH relative labor income shares of the other labor groups $L_j$ decline, when $L_i$ is increased.

The labor services, $(L_1, L_2, ..., L_M)$, can refer to any disaggregation of labor supply. Our services relate to labor age groups; hence CRESH (15) relative wages will represent:

*Age Group Wage Differentials* - to be linked up to demographic labor supply projections.

Since logically, $E \left( \frac{w_i}{w_j}, L_i \right) = E (w_i, L_i) - E (w_j, L_i)$, $E \left( \frac{\varepsilon_i}{\varepsilon_j}, L_i \right) = E (\varepsilon_i, L_i) - E (\varepsilon_j, L_i)$, we should embed the pairwise CRESH relative annual wage relations and ratio elasticities (13-18) into a complete CRESH framework of comparative statics for the absolute ‘own-price’, $E (w_i, L_i)$, ‘cross-price’, $E (w_j, L_i)$, wage elasticities, factor share (distributional) elasticities, and hereto labor substitution and labor complementarity elasticities.

All these elasticities and the basic economic implications of CRESH function (7-8) are revealed and derived below by using duality theory for implicit CRESH Aggregator function, (10): $f (L_1, L_2, ..., L_M)$, Wage Cost function, $C (w_1, w_2, ..., w_M, L_A)$, and Wage Income function, $W (L_1, L_2, ..., L_M, W_A)$. Our new and important expressions for labor complementarity elasticities ($c_{ij}$) are derived for CRESH, (7), (10), in Appendix B.7.

### 3.1 CRESH model calibration and validation: 2010, 2013

In Table 4 (Col. 2, 5, 6c) is collected the 2010 data of wage shares, ($\varepsilon_i$), relative wages, ($w_i/w_j$), $w_i$, $i=1,..,M$. The intensity (weight) parameters ($\alpha_i$) in CRESH Labor supply (7) - are obtained by calibrations, as described below; cf. Guest and Jensen (2016, p.30).

From (15), we get:

$$ \frac{\alpha_i}{\alpha_j} = \frac{\varepsilon_i \rho_j (L_i/L_A)^{\rho_j}}{\varepsilon_j \rho_i (L_i/L_A)^{\rho_i}} = \frac{\varepsilon_i \rho_j (L_j)^{\rho_j}}{\varepsilon_j \rho_i (L_i)^{\rho_i}} L_A^{\rho_i - \rho_j} ; \quad i \neq j \quad (19) $$

In (19), 2010 wage shares ($\varepsilon_i$), $i=1,..,M$, are known (Col.2), and so by making particular assumptions (choices) of the substitution parameters ($\rho_i$), $i=1,..,M$ in Col.3, and by using also the 2010 data, $L_i$, $i=1,..,M$, $L_A = L$, from Table 2 (Col.5), the ratios of the intensity parameters, ($\alpha_i/\alpha_j$), can then be derived (calculated) from the equation (19).

7Ratio elasticity, $E \left( \frac{w_i}{w_j}, L_i \right)$, in (17), comes from (87-88) & complementarity elasticities, $c_{ij}$, (78-79).
### Table 4: Wage income shares, (i), CRESH parameter values, (i), Relative wages, (i), Absolute wages, (i) : Data and Model - Denmark 2010, 2013

<table>
<thead>
<tr>
<th>Age Group</th>
<th>2010</th>
<th>2013</th>
</tr>
</thead>
<tbody>
<tr>
<td>Age</td>
<td>0.0061</td>
<td>0.0316</td>
</tr>
<tr>
<td>Wage share</td>
<td>0.0176</td>
<td>0.0455</td>
</tr>
<tr>
<td>Overall LFP in Table 2</td>
<td>187061</td>
<td>187005</td>
</tr>
<tr>
<td>LFP (2010)</td>
<td>358224</td>
<td>29529</td>
</tr>
<tr>
<td>LFP (2013)</td>
<td>0.0123</td>
<td>0.4432</td>
</tr>
<tr>
<td>in Table 2</td>
<td>0.4572</td>
<td>201755</td>
</tr>
<tr>
<td>Column 1: Labor participation rates (LPR) of age groups - derived from columns (7,10).</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Source:** Year 2010, see *Table 2*; Year 2013: UNITED NATIONS; STATISTICS DENMARK (Department of Labor and Income), Copenhagen.

**Column 1:** Age groups, i = 1, ..., 11; i = 1: 15-19, ..., i = 11: 65-69.

**Column 2:** Age group wage income shares (wage-cost shares in the total wage bill), cf. column 9, Table 2.

**Column 3:** Substitution parameters in CRESH function equation.

**Column 4:** Intensity parameters in CRESH function equations, L\(_i\) = L = 211472.

**Column 5:** Actual relative wages (Age group annual wage procile), given by data, Table 2.

**Column 6a:** CRESH Model (optimal) Relative annual wages (Age group annual wage profile), 2010; \( r = 4.378, L_{L} = L = 2112472 \).

**Column 6b:** CRESH Model Absolute annual wages (Age groups), with, W (2010) = 437552, and with calculated \( w_{i} (2010) = 4.10492 = (w_{i} \text{ tilde}) \).

**Column 6c:** The observed Annual wages 2010 : Data, cf. column 6, Table 2.

**Column 6d:** CRESH Model Absolute annual wages (Age groups) with, W (2010) = 437552, and with calculated \( w_{i} (2010) = 4.10492 = (w_{i} \text{ tilde}) \).

**Column 7:** CRESH Model Absolute annual wages (Age groups) with, W (2010) = 437552, and with calculated \( w_{i} (2010) = 4.10492 = (w_{i} \text{ tilde}) \).

**Column 8:** CRESH Model Absolute annual wages (Age groups) with, W (2010) = 437552, and with calculated \( w_{i} (2010) = 4.10492 = (w_{i} \text{ tilde}) \).

**Column 9:** CRESH Model Absolute annual wages (Age groups), with, W (2010) = 437552, and with calculated \( w_{i} (2010) = 4.10492 = (w_{i} \text{ tilde}) \).

**Column 10:** CRESH Model Absolute annual wages (Age groups), with, W (2010) = 437552, and with calculated \( w_{i} (2010) = 4.10492 = (w_{i} \text{ tilde}) \).

**Column 11:** CRESH Model Absolute annual wages (Age groups), with, W (2010) = 437552, and with calculated \( w_{i} (2010) = 4.10492 = (w_{i} \text{ tilde}) \).

**Column 12:** CRESH Model Absolute annual wages (Age groups), with, W (2010) = 437552, and with calculated \( w_{i} (2010) = 4.10492 = (w_{i} \text{ tilde}) \).

**Column 13:** CRESH Model Absolute annual wages (Age groups), with, W (2010) = 437552, and with calculated \( w_{i} (2010) = 4.10492 = (w_{i} \text{ tilde}) \).

**Column 14:** CRESH Model Absolute annual wages (Age groups), with, W (2010) = 437552, and with calculated \( w_{i} (2010) = 4.10492 = (w_{i} \text{ tilde}) \).

**Column 15:** CRESH Model Absolute annual wages (Age groups), with, W (2010) = 437552, and with calculated \( w_{i} (2010) = 4.10492 = (w_{i} \text{ tilde}) \).

**Column 16:** CRESH Model Absolute annual wages (Age groups), with, W (2010) = 437552, and with calculated \( w_{i} (2010) = 4.10492 = (w_{i} \text{ tilde}) \).

**Remark:** As mentioned in Note 5, both the Overall LFP and some of the Age-specific LFP, \( L = L_{L}/N \), in *Table 2 (A)* (column 10,16) are smaller than often reported as LFP rates.

Labor Force in LFP (ILO statistics) consists of: Employees, and Unemployed. \( L_{L} \) in *Table 2 (A)* (column 5,10), are Full-time equivalents. Unemployment rate, 2010-13: 6%, (approx. 100,000 persons).
By next summing the equation (19), and using, \( \sum_{i=1}^{M} \alpha_i = 1 \), cf. (8), we have,

\[
\sum_{i=1}^{M} \frac{\alpha_i}{\alpha_j} = \frac{1}{\alpha_j} = \sum_{i=1}^{M} \frac{\varepsilon_i \rho_j (L_i/L_A)^{\rho_i}}{\varepsilon_j \rho_i (L_i/L_A)^{\rho_i}} = \frac{\rho_j}{\varepsilon_j} \left[ \frac{L_j}{L_A} \right]^{\rho_j} \sum_{i=1}^{M} \frac{\varepsilon_i}{\rho_i} (L_i/L_A)^{\rho_i} = \frac{\rho_j}{\varepsilon_j} \left[ \frac{L_j}{L_A} \right]^{\rho_j} \mu \quad (20)
\]

Thus the size of \( \alpha_j \) is determined by the RHS expression of (20). Next (19-20) give:

\[
\alpha_i = \frac{\varepsilon_i \rho_j (L_j/L_A)^{\rho_i}}{\varepsilon_j \rho_i (L_i/L_A)^{\rho_i}} \quad \alpha_j = \frac{\varepsilon_i}{\rho_i} \left[ \frac{L_A}{L_j} \right]^{\rho_i} \frac{1}{\mu}, \quad i = 1, \ldots, M \quad (21)
\]

Hence all absolute values of \( \alpha_i \) parameters are obtained by (21) - [with \( \mu \) as seen in (20)].

By this calibration procedure, (21), such associated values of 11 CRESH intensity parameters, \( (\alpha_i) \), Table 4, (Col.4), for 2010 can - with \( L_i \), \( i=1,\ldots,M \), \( L_A = L \), from Table 2 - be calculated for any assumptions (pattern) of these 11 parameters, \( \rho_i \), Col.3.

The actual selected CRESH substitution parameters \( \rho_i \) in Table 4, (Col.3) were determined as follows. An initial set of 11 \((\rho_i)\) determines 11 \((\alpha_i)\), as described by (19) and (21). The aim is to find a pattern of \( \rho_i \) which generates CRESH relative wages (15) by \((\frac{w_i}{w_j})\), \( j=4 \), (22) that best fit, (Col.6a), the actual relative wages, (2010), \((\frac{w_i}{w_j})\), (Col.5).

\[
\frac{w_i}{w_4} = \frac{\alpha_i \rho_i}{\alpha_4 \rho_4} \frac{L_i^{\rho_i-1}}{L_4^{\rho_4-1}} L_A^{\rho_4-\rho_i}, \quad i = 1, \ldots, M \quad \leftrightarrow \quad \frac{w_i}{w_4} = \frac{\alpha_i \rho_i}{\alpha_4 \rho_4} \lambda_i^{\rho_i-1} \lambda_4^{\rho_4-1} \quad (22)
\]

Various patterns of \( \rho_i \) have been tested in this way for Australia as discussed in Guest & Jensen (2016). The best fit for Denmark is found to be the approximate U-shape pattern of \( \rho_i \), shown in Table 4, (Col.3). This U-shape pattern of \( \rho_i \) implies that middle age workers, who have relatively low values for \( \rho_i \) have a mix of labor attributes (‘qualities’) that make them harder to substitute (replace) [lower \( \rho_i \) give smaller substitution elasticities, \( \sigma_{ij} \), (49)] than the younger or older workers. This pattern has also important consequences for wages, relative and absolute [lower \( \rho_i \) give larger labor complementarity elasticities, \( c_{ij} \), (57), for age group \( i \)], and so group \( i \) have larger annual wage elasticities, \( E(w_i, L_i) = \varepsilon_j c_{ij} \), (86), and hence gain larger wage increases by bigger labor supplies of other age groups \( L_j \). The combined set (sizes) for \( \rho_i, \alpha_i \), Col.3-4 fitted best 2010: (Col.5, 6a); or (Col.6b, 6c) by (28-29), \( \lambda_i \) (22), RHS, cf. footnotes 8-9 below.

To validate the calibrated year 2010 CRESH parameter values, \((\rho_i, \alpha_i)\), Table 4, Col.3-4, we corroborate these parameter sizes \((\rho_i, \alpha_i)\) upon another data set, year 2013. Thus the 2013 data seen in the six columns, Table 4, Col.7-12, correspond (with same content/explanations) exactly to earlier six columns for year 2010, Table 2, Col.2-7.
In order to validate the calibrated parameters outside the base year (2010), we insert the calibrated (2010) values, $\rho_i$, $\alpha_i$, (Col.3-4), into our formula of relative wage, (22), together with using the observed 2013 data for: $L_i, i=1,..,M, L_A = L$, (Col.10). Thus columns (Col.3-4,10) give by (22) the CRESH results for relative wages, $\frac{w_i}{w_L}^j$, (j=4), Col.13 for year 2013 to be compared with observed relative wages, $\frac{w_i}{w_L}$, (j=4), Col.12 for 2013. Apart from age groups (60-64, 65-69), the Col.12-13 are concuring pretty well for all age groups. So the calibrated CRESH parameters, $\rho_i$, $\alpha_i$, (Col.3-4), are essentially confirmed (validated) on the new data set (2013).

By (22), only the relative wage numbers ($\frac{w_i}{w_L}$) were calculated for 2010 og 2013. But how to get the absolute sizes of the Annual wages ($w_i$) for Ages, $i=1,..,M$, in Table 4?

**Absolute wages.** Total Wage Income for all Age groups ($i$) is by definition, $wL$, i.e,

$$wL = \sum_{i=1}^{M} w_i L_i; \quad w_4 = \frac{wL}{L_4 + \sum_{i\neq 4}^{M} \frac{w_i}{w_L} L_i}; \quad w_i = w_4 (2013) \cdot \frac{\alpha_i \rho_i (L_i^{\rho_i - 1})}{\alpha_4 \rho_4 (L_4^{\rho_4 - 1})} \frac{L_A^{\rho_A - \rho_i}}{L_4} \quad (23)$$

Dividing LHS of (23) by $w_4$, and using $[\frac{w_i}{w_L}], i \neq 4$, rearranging, gives $w_4$, stated above. With $w_4$ (23) allows all $w_i$ for 2013 to be calculated by RHS (23), by using observed, $w_4 (2013) = 423743$, Table 4 (Col.14). However, by $w_4 (2013)$ as ‘scaling factor’ for $w_i$, (23), generates for $L_A = L (2013) = 2108014$, the Total Wages: $wL$ (2013) =930142 Billion, Average Annual wage, $w$ (2013) = 441241. But Col.11, 15, give for $L_A = L (2013)$ the actual Total Wage sum: $wL$ (2013) = 968555 Billion, Average Annual wage, $w$ (2013) = 459463 $\equiv W_A (2013)$, i.e., $W_A (2013)$ gives by $w_4 = \bar{w}_4$ as in (24) a consistent scaling wage of $\frac{w_i}{w_L}$ to use in computing absolute annual wages ($w_i$) for age, $i=1,..,M$, cf. Col.15:

$$w_i = \bar{w}_4 (2013) \cdot \frac{\alpha_i \rho_i (L_i^{\rho_i - 1})}{\alpha_4 \rho_4 (L_4^{\rho_4 - 1})} \frac{L_A^{\rho_A - \rho_i}}{L_4} , \quad i=1,..,M; \quad \bar{w}_4 (2013) = \frac{W_A(2013)}{L_A} = \frac{L}{L_A + \sum_{i\neq 4}^{M} \frac{w_i}{w_L} L_i} \quad (24)$$

All $w_i$ (24) were still calculated with chosen $\alpha_i$ and $\rho_i$ parameters from 2010, Table 4. By using (22), (24), we have a consistent CRESH formula for absolute Age wage ($w_i$) calculations to be applied any year ($t$) - also with $W_A$, $w_i$, defined by (25), (26) below.

\(^\text{8}\)CRESH $\gamma$, (7), a "total productivity" (efficiency) parameter was not involved in relative wages, (15).

For given values of $\alpha_i$, $\rho_i$, Table 4 (Col.3-4), the $\gamma$ size can be adapted so that aggregate variable, (7), $L_A = L$ (Total Labor force, Labor supply) = $L(t) = \sum_{i=1}^{M} L_i$; for $t$=2010, 2013, see $\gamma$, Table 4 (Col.6,13).

Such $\gamma$ values are to be used for any year, if as in all Tables 4-7, $L_A = L(t) = \sum_{i=1}^{M} L_i (t)$. 19
3.1.1. From dual CRESH Labor Cost function (52), or dual \textbf{Wage Income} function (81) in \textbf{App.B}, we have for CRESH - $F(L_A, L_1, L_2, \ldots, L_M) = 0$, (7); $L_A = f(L_1, L_2, \ldots, L_M)$, \textbf{Aggregator} (10) - the \textit{basic duality} relations, cf. (12), (14-15), (52), (67-68), (81), (84):

\[ W_A L_A = \sum_{i=1}^{M} w_i L_i \equiv w L \equiv W \equiv C \equiv c(w_1, w_2, \ldots, w_M) L_A; \quad W_A = c(w_1, w_2, \ldots, w_M) \quad (25) \]

\[ w_i = w_i(L_1, L_2, \ldots, L_M, W_A) = \frac{\partial W(L_1, L_2, \ldots, L_M, W_A)}{\partial L_i} = W_A \frac{\partial L_A(L_1, L_2, \ldots, L_M)}{\partial L_i} \quad (26) \]

\[ W_A \frac{\partial f(L_1, L_2, \ldots, L_M)}{\partial L_i} = W_A \frac{\alpha_i \rho_i (L_i/L_A)^{\rho_i-1}}{\sum_{i=1}^{M} \alpha_i \rho_i (L_i/L_A)^{\rho_i}} = W_A \frac{\alpha_i \rho_i \lambda_i^{\rho_i-1}}{\sum_{i=1}^{M} \alpha_i \rho_i \lambda_i^{\rho_i}} \quad (27) \]

\textbf{W}_A (25): \textit{Arithmetic Average} of all \textit{money age wages} ($w_i$) - or 'shadow values', (26-27).

For demographic projection period, $t: \text{2015-2090}$, we don’t have - as (24) in Table 4 - empirical values of $\textbf{W}_A (t)$. Throughout the projection period the \textit{exogenously} imputed size to $\textbf{W}_A$ (25-27) is $\textbf{W}_A$ (2010), \textbf{Tables (2,3)}. Thus our \textbf{absolute} Annual wages are:

\[ w_i = \bar{w}_4 \cdot \frac{\alpha_i \rho_i (L_i)^{\rho_i-1}}{\alpha_4 \rho_4 (L_4)^{\rho_4-1} L_A^{\rho_4-\rho_i}}; \quad \bar{w}_4 = \frac{W_A(2010) L}{L_4 + \sum_{i \neq 4} \frac{w_i}{L_i}} \quad ; \quad i = 1, \ldots, M \quad (28) \]

By our \textit{$\gamma$ calibration}^{9}, hence $\lambda_i \equiv \frac{L_i}{L_A}; \sum_{i=1}^{M} \lambda_i = 1, w_i (28)$ is equivalent to, cf. (22-24), (5):

\[ w_i = \bar{w}_4 \cdot \frac{\alpha_i \rho_i}{\alpha_4 \rho_4} \lambda_i^{\rho_i-1}; \quad \bar{w}_4 = \frac{W_A(2010)}{L_4 + \sum_{i \neq 4} \frac{w_i}{L_i}} \lambda_i \equiv \frac{L_i}{L_4} \frac{n_i}{l_{15-69} n_{15-69}}; \quad i = 1, \ldots, M \quad (29) \]

Note that (26-29) give the same wages $w_i$, but CRESH \textit{duality} formulas (25-27) provide economic content and intuition. We saw an illustration of (28-29) in Table 4, (Col.6b). For $\textbf{W}_A$ (2010) = 437552 and, $L_A = L(2010) = \sum_{i=1}^{M} L_4 (2010)$, with all $L_i (2010)$ in Table 2, (Col.5), the calculation of $\bar{w}_4$ by (28-29) gives, $\bar{w}_4 (2010) = \textbf{410492}$; applying this $\bar{w}_4$ as 'scaling multiplier' to all wage ratios, ($\frac{w_i}{\bar{w}_4}$), Table 4 (Col.6a), gives CRESH \textbf{absolute} (money) \textbf{annual wages}, $w_i$ (2010) (Col.6b) - actual observed ($w_i$), \textbf{data} are in (Col.6c).

Finally, note CRESH formulas (26-29) in 2010 give \textit{higher wages} for $w_{60-64}$, $w_{65-69}$ than to $w_{55-59}$, $w_{45-49}$ (despite \textit{lower substitution parameters} : $\rho_{55-59}$, $\rho_{45-49}$, (Col.5). The influence of much \textit{smaller} Labor supplies (scarcity) of $L_{60-64}$, $L_{65-69}$, Table 2 (Col.5), \textit{dominate} (22), (27-28), and explain the \textit{high}, $w_{60-64}$, $w_{65-69}$, in both model/data 2010.

\textbf{CRESH Age wage profiles}, (26-29), of the age-groups \textit{over time} are complex, but versatile - as will be seen in projected \textit{calendar years}, and over entire \textit{cohort life times}.

\textsuperscript{9}See footnote 8 - where for year 2010 : $\gamma = 4.378$. Using (29), $\gamma$'s to \textbf{Tables 5-7} are not needed.
3.2 Disaggregations of Labor Supply - CRESH Subaggregators

The Register based Columns (5-6), Table 2, of age-specific (labor, wage) data, \((L_i, w_i)\), \(i=1,..,11\), [making 75% of GDP (Value Added), \(wL/Y = 924.3/1228.1 = 0.75\), cf. Table 3] form directly by Column (7) empirical points outlining a shape, seen below in Fig. 2d.

The Age-wage profile in Columns (6-7) refers to the complete Danish Labor supply (age 15-69), year 2010 (in full time equivalents): men, women, every occupation, private and public sector, all lengths of schooling, educations, etc. 10

Standard Human capital (Labor quality levels) models posit that earnings (wages) rise with levels (years) of Schooling (5-7, 9-11, 13-15), or with Education levels (High school, College, Graduate school), or with Occupational classifications [blue-collar (skilled/craftsman, unskilled) workers, white-collar (professionals, administrators, clerical) employees].

If available data of age-specific Labor inputs and wages, \((L_i, w_i)\), are disaggregated (by subscript): \((L_{iJ}, w_{iJ})\), into e.g., 8 quality levels \((J)\), we may construct 8 CRESH Subaggregators, \(L_{A_j} = f_j(L_{1j}, L_{2j}, \ldots, L_{Mj})\), cf. (10), and hence analogous to (15) get wage ratios, \(\frac{w_{iJ}}{w_{jJ}}\); by analogous duals of, (25-27), \(W_{A_j} L_{A_j} = \sum_{i=1}^{M} w_{iJ} L_{iJ} \equiv w_j L_j \equiv W_j\), the money wages \((w_{iJ})\) of ages and qualities of Labor input/supply \((L_{iJ})\) become:

\[
  w_{iJ} (t) = W_{A_j} \frac{\partial f_j(L_{1j}, L_{2j}, \ldots, L_{Mj})}{\partial L_{iJ}} = W_{A_j} \sum_{i=1}^{M} \frac{\alpha_{iJ} \rho_{iJ} \lambda_{iJ}^{i-1}}{\sum_{i=1}^{M} \alpha_{iJ} \rho_{iJ} \lambda_{iJ}^{i-1}}, \quad i = 1, \ldots, M, \quad J = I, II, \ldots, VIII \tag{30}
\]

For each year \((t)\), disaggregated data \((L_{iJ}, w_{iJ})\) can for each level \((J)\) be organized by age \((i)\) as in Table 2, Col.(5-6), and the analogous CRESH wage formulas, (29), for each level \((J)\) can be implemented for \(w_{iJ} (t)\), (30), as in Table 4, Col.(6a, 6b, 6c), (11,13,15).

Estimating different Age-wage profiles, \(w_{iJ}, \quad i=1,\ldots,M, \quad (30)\), corresponding to each school level \((J)\), Hanoch (1967, p.315-319) obtained 8 Age-wage profiles of essentially similar shape, but stacked vertically above each other with higher school level \((J)\).

Although Hanoch (1967) did not formally use Labor subaggregator functions, but vertical

---

10The high wages of two age groups, (60-64, 65-69), in the Danish Age-wage profiles, Fig. (2d, 2e), Fig. (12, 13), a puzzle, are to some extent, partly due to their high proportions of Public Sector employees with seigniory wage systems (Medical profession in Public Hospitals, other Academics in Government Services (including Universities, Secondary Schools). Lower paid Public Sector employees in Primary School and Hospitals have mostly retired by age 65 in 2010 - as in the Private Sector.

Age-wage profiles on Disaggregated Labor data should re-establish global concavity of age-wage profiles.
shifting\textsuperscript{11} on disaggregated data - of age-wage profiles by a specific level (exogenous) variable is seen as an extension of parametric CRESH age-wage formula (27), where \( W_A \) (26) is an exogenous vertical shift variable of all (one quality level) wages, \( w_i, i=1,...,M \). Thus all disaggregated CRESH Age-wage profiles, \( w_{i,t} (t) \), (30), are for any year (t) stacked vertically\textsuperscript{12} through their quality level Average wage, \( W_A, t \), cf. (25).

To see clearly how Subaggregators, \( L_{A,i} = f_j(L_{i,j}, L_{i,j}, ..., L_{M,j}) \), work in (30) without attention to Total Labor supply numbers (\( L_{A,i} \), we may recall that CRESH function, (7), is homogeneous of degree zero: \( F(L_{A}, L_{1}, L_{2}, ..., L_{M}) = F(1, \lambda_1, \lambda_2, ..., \lambda_M) \), and that Aggregator, \( L_A = f (L_1, L_2, ..., L_M) \), (10), Subaggregators, are homogeneous of degree one - implying that all their partial derivatives: \( \frac{\partial f_j}{\partial L_{i,j}} \), (31) - are homogeneous functions of degree zero, such that as stated in (12), (30), we have partial derivatives:

\[
\frac{\partial f_j}{\partial L_{i,j}}(\lambda_{i,j}, \lambda_{i,j}, ..., \lambda_{M,j}) = \alpha_{i,j} \rho_{i,j} \lambda_{i,j}^{\rho_{i,j}-1} \text{, } i = 1, M, J = I, .., VIII (31)
\]

By CES (9), the derivatives (31) are much simplified, as denominator above drops out:

\[
\forall i: \rho_{i,j} = \rho, \frac{\partial f_j}{\partial \lambda_{i,j}}(\lambda_{i,j}, \lambda_{i,j}, ..., \lambda_{M,j}) = \alpha_{i,j} \lambda_{i,j}^{\rho_{i,j}-1} \text{, } i = 1, M, J = I, .., VIII : \sum_{i=1}^{M} \alpha_{i,j} \lambda_{i,j}^{\rho_{i,j}} = 1 (32)
\]

Only derivatives (31-32) of Aggregator functions, \( L_{A,j} = f_j \), are used in imputing wages \( w_{i,j} \), (30), to the age-groups (i) of Total Labor supply, \( L_{A,i} = \sum_{i=1}^{M} L_{i,j} \).

The derivatives (27), (31-32) are not marginal products (output) of \( L_i \) in age group (i), but marginal contributions of \( L_i \) to \( L_A \) by the Aggregator function, \( \frac{\partial L_A}{\partial L_i} = \frac{\partial f_j}{\partial L_{i,j}} \), cf. (12); marginal contributions, \( \frac{\partial L_A}{\partial L_i} \), do not depend on (invariant to) the absolute sizes of \( \{L_A, L_i\} \), but only upon the size of \( \lambda_i \), in CES, (32) - and upon all \( \lambda_i \) with CRESH, (31).

As in (15) efficient utilization of Labor supplies - within \( L_{A,j} = \sum_{i=1}^{M} L_{i,j}, J = I, II \) - requires that the ratio (relative) of age-wages were equated to the ratio (relative) of their marginal contribution: \( \frac{w_{i,j}}{w_{j,j}} = \frac{\partial f_j(\lambda_{i,j}, \lambda_{j,j}, ..., \lambda_{M,j})}{\partial \lambda_{i,j}}/\frac{\partial f_j(\lambda_{i,j}, \lambda_{j,j}, ..., \lambda_{M,j})}{\partial \lambda_{j,j}}, i \neq j \). With data and the accounting identities, \( W_{A,j}, L_{A,j} = \sum_{i=1}^{M} w_{i,j} L_{i,j} = W_j, J = I, .., VIII \), we have the Average wages: \( W_{A,j} = W_j/L_{A,j}, J = I, .., VIII \). Thus, by (31) and, \( W_{A,j} \), we have also the absolute money wages \( w_{i,j} \) for all Age groups (M) in all the Labor categories (qualities), (VIII):

\[
w_{i,j}(t) = W_{A,j} \alpha_{i,j} \rho_{i,j} \lambda_{i,j}^{\rho_{i,j}-1} \text{, } i = 1, M, J = I, .., VIII (33)
\]

which are the CRESH Calendar year (t) Age-wage profiles, (30), , restated in \( \lambda_{i,j} \).

\textsuperscript{11}The disaggregated age-wage profiles not only shift vertically, but they may also twist/rotate.

\textsuperscript{12}Age-earnings (wage) profiles from education have a long economic history, Blaug (1967, p.337).
Thus analytic wage structure description for different Labor qualities (education levels) require the analytic tools of CRESH Subaggregator, \( L_{ij} = f_i(L_{1i}, L_{2j}, \ldots, L_{Mj}) \), derivatives (31) as used in (33)\(^{13}\). See hereto the Canonical Model in Appendix C. Finally, we note that changes in wage structure (distributions) can be analyzed in calendar years (section 4) by the apparatus of CRESH Labor Aggregators [not using solely total Labor supplies, but only age distributions, \( \lambda_{ij}, i = 1, \ldots, M \)] - without production functions.

### 3.3 Outputs and Multi-factor CRESH Production Functions

For a long time, the scope of Macro (Y) models has been enlarged by increasing the number of primary factors. But here a problem has also existed for years, viz. that with more than two factors the multi-factor CES function has the same constant substitution elasticity (\( \sigma \)) between any and all factors - severe restriction that we removed by CRESH Labor aggregator, \( F(L_A, L_1, L_2, \ldots, L_M) = 0, (7), (10) \), and the Sub-aggregators above.

The CRESH functional form can also be used to CRESH implicit production functions:

\[
G(Y, X_I, X_{II}, \ldots, X_V) = G(Y, L_I, L_{II}, K_{III}, K_{IV}, K_V) = \gamma \sum_{j=I}^{V} \alpha_j \left( \frac{X_j}{Y} \right)^{\rho_j} - 1 = 0 \quad (34)
\]

\[
\gamma > 0; \quad \forall j: \alpha_j > 0, \quad \sum_{j=I}^{V} \alpha_j = 1; \quad \forall j: \ 0 < \rho_j \leq 1 \text{ or } \rho_j < 0 \quad (35)
\]

where the parameters (35) again preserve the important global regularity properties. As in (10), a unique implicit production function, \( Y = g(X_I, X_{II}, \ldots, X_V) \) exists and \( g \)

\[
\forall X_j > 0 : \ Y = g(X_I, X_{II}, \ldots, X_V) > 0 ; \quad \frac{\partial g}{\partial X_j} > 0, \quad \frac{\partial^2 g}{\partial X_j^2} < 0 ; \quad Y = \sum_{j=I}^{V} \frac{\partial g}{\partial X_j} X_j \quad (36)
\]

having all the globally regularity properties as the Labor Aggregator, \( L_A = f, (10) \). All expressions and illustrations of the Substitution elasticities and the Complementarity elasticities in Appendix A-B carry over to (34-36).

Old problems with different substitution elasticities between two Labor categories, \( L_i, L_{II} \), and various nonlabor inputs such as services of Capital goods\(^{14}\), (34), can be

\(^{13}\)Labor Aggregator derivatives, (27), (30), (33), are analogous to 'Inverse factor (consumer) demand functions' by derivatives of production (utility) functions; first-order and second-order derivatives define complementarity elasticities, \( c_{ij}, (56), (83) \), giving wage elasticities, (63), (86-88), w.r.t Labor supplies.

resolved with proper Macro wage numbers assigned to $W_{AJ}, J = I, II$ - and subsequently used for the Age-wage profiles of the two Labor Subaggregates, $w_{1J}$, in (33-34).

Analogously to (27), Macro money wages $W_{AJ}, J = I, II$, are simply derived from CRESH macro production function, (34-36) (single output, $Y$) and the output price ($P$):

$$W_{AI} = P \cdot \frac{\partial g(L_1, L_{II}, K_{III}, K_{IV}, K_V)}{\partial L_I} = P \cdot \frac{\alpha_I \rho_I (L_1/Y)^{\rho_I-1}}{\sum_{J=1}^{J=M} \alpha_J \rho_J (X_J/Y)^{\rho_J}}$$  \hspace{1cm} (37)

$$W_{AII} = P \cdot \frac{\partial g(L_1, L_{II}, K_{III}, K_{IV}, K_V)}{\partial L_{II}} = P \cdot \frac{\alpha_I \rho_I (L_1/Y)^{\rho_I-1}}{\sum_{J=1}^{J=M} \alpha_J \rho_J (X_J/Y)^{\rho_J}}$$  \hspace{1cm} (38)

Depending on the evolution (time series) of factor productivities (unit requirements), $(L_1/Y, L_{II}/Y, K_{III}/Y, K_{IV}/Y, K_V/Y)$, the sizes of the two Macro wages (37), (38), are changing, which shift the Calendar year Subaggregate (Micro) Age-wage profiles, (33). Shifting of (33) by $W_{AJ}$, (37-38), does not alter the shape of (33) and its relative wages.

### 3.3.1. Inverse Labor demands - Wage functions - Age-wage profiles, and Empiric methods

Standard labor demand analyses have estimated various explicit production functions, $Y = G(L_1, L_{II}, L_{IV}, L_V, K_I, K_{II}),$ as e.g., Trans-Log, Freeman (1979), cf. Introduction, Hamermesh & Grant (1979, p.538; 1981, p.357), or Generalized Leontief production function, Borjas (1986, p.59), to obtain relevant Labor demand functions, complementarity elasticities and partial wage elasticities - survey in Hamermesh (1993). In Multi-factor Production functions, many classifications into Labor & wage sub-groups were used: various occupations, educations (length of schooling), gender (male, female), age (young, middle age, old). However, we are not using production functions at all; we have no proper data for capital inputs (quantities or their factor prices). Instead, we have for our purposes a complete data set of Danish Labor supplies and wages, seen in sections 2.2, 3.1. Hence we used (constructed) and estimated (calibrated) the CRESH Labor Aggregator function, $L_A = f(L_1, L_2, ..., L_M), (7-8), (10)$, and accordingly here get its Inverse Labor demand system as, $w_i = W_A \cdot \frac{\partial f(L_1, ..., L_M)}{\partial L_i}, i = 1, 2, ..., M, \text{ cf. (92)}$, or as wage functions also called Age-Annual Wage profiles, which in explicit parametric CRESH form is stated in the equation, (27). The CRESH Age-Annual wage formula (27) is used in sections 4-5 to perform analytic 'controlled experiments' of Demographic impacts upon Calendar year wages and Cohort life cycle wages. In these scenarios, the benchmark value of $W_A$ is $W_A(2010)$ - and hence (27) becomes 'operative' as (28-29).
4 Demographics, Labor supplies, and Calendar wages

4.1 Projected labor age groups, relative wages, annual wages

Danish Population sizes, \( N_i(t) \), for the 11 age-groups (i) of working life (15-69) - obtained from United Nations (2015) source, cf. Table 1 - are seen in Tables (5a, 5b), Col.1. Danish Labor Supplies (full-time workers), \( L_i(t) \), (39), in age-groups (i) - calculated by \( N_i(t) \), (4), and Labor Participation rates, \( l_i \) (2010), Table 2 - are Tables (5a, 5b), Col.2,

\[ L_i(t) = l_i(2010) \cdot N_i(t), \quad t = 2020, 2030, 2040, 2050, 2070, 2090; \quad L(t) = \sum_{i=1}^{M} L_i(t) \quad (39) \]

e.g., \( L_{15-19}(2020) = 0.0855 \cdot 338740 = 28962, \quad L_{35-39}(2030) = 0.7468 \cdot 412710 = 308212. \)

The Participation rates as \( l_i \) (2010) are held constant through the whole demographic projection period (2020-2090), and for all (Medium, Low, High) demographic variants. In the three Fertility variants described in Table 1a, the Fertility change commences in 2015, cf. Fig.1. This implies that Labor Supplies, \( L_i(t) \), starting \( i = 15 \) - are equal for all fertility variants until 2030; hence 2035 is the first five year period in which labor supplies, \( L_i(t) \) differ across the three fertility variants. Hence we report population, labor supplies for 2020 and 2030 separately in Table 5a, since these are common to all variants. But age-specific wages \( w_i(t) \) are not constant for 2020, 2030, as they have different \( L_i(t) \).

Table 5b extends Table 5a for 2040 to 2090 for the three fertility variants. In Tables (5a, 5b), last Col. are shown in all years/variants the Age wage profile of 2010, \( w_i(2010) \). The relative age-group wages, \( w_i(t)/w_{4}(t) \) - calculated by inserting \( L_i(t) \) and total \( L(t) \) from (39) into CRESH, (22), with \( L(t) = L_A \) - are exhibited in Tables (5a, 5b), Col.3.

\[
\frac{w_i(t)}{w_{4}(t)} = \frac{\alpha_i \rho_i}{\alpha_4 \rho_4} \frac{L_i(t)^{\mu_i-1}}{L_4(t)^{\mu_4-1}} L(t)^{\mu_4-\mu_i}, \quad t = 2020, 2030, 2040, 2050, 2070, 2090 \quad (40)
\]

The values for \( w_i(t)/w_{4}(t) \) in Table 5b differ for each variant, as \( L_i(t) \) (column 2) differ.

The conforming absolute/money age-group wages, \( w_i(t) \) - by multiplying (40) with money wage, \( \bar{w}_i(t) \), with \( W_A(2010) = 437552 \), cf. (28), (84) - are in Tables (5a, 5b), Col.4:

\[
w_i(t) = \bar{w}_4(t) \cdot \frac{\alpha_i \rho_i}{\alpha_4 \rho_4} \frac{L_i(t)^{\mu_i-1}}{L_4(t)^{\mu_4-1}} L(t)^{\mu_4-\mu_i}, \quad i = 1, \ldots, 11; \quad \bar{w}_4(t) = \frac{W_A(2010) L(t)}{L_4(t) + \sum_{i \neq 4}^{11} \frac{w_i(t)}{w_{4}(t)} L_i(t)} \quad (41)
\]

Thus Tables (5a, 5b) present 'comparative' age-group annual wages in two forms: directly as ratio: \( \frac{w_i(t)}{w_{4}(t)} \), (40), and on absolute income scale as: money wage, \( w_i(t) \), (41).
Both these two forms (40-41) are necessary for calculating and understanding the influence of demographic projections $N_i(t)$ via $L_i(t)$, (39), (4), upon any and all (11) 'age group (i) annual wages', $w_i(t)$, (41), (29) - Age-wage profile - in Calendar years (t), Tables (5a, 5b).¹⁵ Later translated $w_i(t)$ of (41) are used in summing annual wages of Cohorts, (43-44), Table 7, during their whole working life (employment years, $i: 15-69$).

<table>
<thead>
<tr>
<th>Age (i)</th>
<th>$N_i$</th>
<th>$L_i$</th>
<th>$w_i/w_{L_i}$</th>
<th>$w_i$</th>
<th>$w_i(2010)$</th>
</tr>
</thead>
<tbody>
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<tr>
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<td>437552</td>
</tr>
</tbody>
</table>

Source: See Table 5b.


The pure impact of increased Life Expectancy, cf. Table 1a, upon Population numbers $N_i$ may noted by comparing for $t=2020$, $t=2030$, the corresponding sizes of the age group of same birth years - e.g., $N_{25-29}$ (2030) = 365930 > $N_{15-19}$ (2020) = 338740 ; $N_{35-39}$ (2030) = 412710 > $N_{25-29}$ (2020) = 398400.

The so-called Millennial Generation (Y), (1981-1995), is seen, $t = 2020$ as: $N_{35-39} + N_{30-34} + N_{25-29}$. The Generation (Z), (1996-2010), is seen above, $t = 2030$ as: $N_{30-34} + N_{25-29} + N_{20-24}$. 

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<table>
<thead>
<tr>
<th>Age (i)</th>
<th>N_i</th>
<th>L_i</th>
<th>w_i</th>
<th>N_j</th>
<th>L_j</th>
<th>w_j</th>
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<td>345.740</td>
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Source: N_i, United Nations (2015), cf. Tables (1, 2); L_i = f (2010) \cdot N_i(t), (39), (39), w_i(t) = \frac{w_i(t)}{w_i(40)}; w_i(t), (41).
Table 5c. Labor supplies and Annual wages of Younger (30-34), Middle aged (45-49), and Older (55-59) workers in Calendar years: 2020, 2030, 2040, 2050, 2070, 2090 – Three Fertility (M, L, H) variants.

<table>
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<th>Year</th>
<th>$L_{30-34}$</th>
<th>$w_{30-34}$</th>
<th>$L_{45-49}$</th>
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<td>409251</td>
<td>282547</td>
<td>490917</td>
<td>265371</td>
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<td>239490</td>
<td>530442</td>
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<tr>
<td>2040</td>
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<td>310995</td>
<td>467867</td>
<td>217355</td>
<td>478566</td>
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<td></td>
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</tr>
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<td></td>
<td>H 268424</td>
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<td>310995</td>
<td>470949</td>
<td>217355</td>
<td>480782</td>
</tr>
<tr>
<td>2050</td>
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<td>409479</td>
<td>287586</td>
<td>464833</td>
<td>244016</td>
<td>444392</td>
</tr>
<tr>
<td></td>
<td>L 222601</td>
<td>417453</td>
<td>287586</td>
<td>477552</td>
<td>283113</td>
<td>436048</td>
</tr>
<tr>
<td></td>
<td>H 283127</td>
<td>403417</td>
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<td>401926</td>
<td>292005</td>
<td>488720</td>
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<td></td>
<td>L 204560</td>
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<td>213658</td>
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<td>493656</td>
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<td></td>
<td>H 403181</td>
<td>399510</td>
<td>346285</td>
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<td>401696</td>
<td>237725</td>
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<td>210878</td>
<td>434898</td>
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<td>H 403181</td>
<td>399510</td>
<td>346285</td>
<td>506101</td>
<td>338346</td>
<td>465578</td>
</tr>
</tbody>
</table>

Source: Tables (5a,5b), rows, age (i): 30-34, 45-49, 55-59.

Based on Table 5c we show in Figures (2a-2c) the annual wages $w_i(t)$ for younger (30-34 years), middle aged (45-49 years) and older (55-59 years) workers in each of the calendar years 2020, 2030, 2040, 2050, 2070, 2090, assuming three different fertility levels.

First, it is possible from Table 5c, Fig. (2a-2c), to get an impression of the importance for the annual wages of belonging to a small (respectively big) Birth group (Generation), Tables (5a, 5b) and hence their subsequent Labor supplies in Table 5c. The clearest example of this is found by looking at $w_{30-34}$, Fig. 2a, of the 30-34 years old in 2020, $L_{30-34}$, born as a very small generation in 1986-1990, next at the highest $w_{45-49}$, Fig. 2b, of the 45-49 years old in 2030, $L_{45-49}$, born as the even smaller generation in 1981-1985, and finally at the high $w_{55-59}$, Fig. 2c, of the 55-59 years old in 2040, $L_{55-59}$, born also in same years, 1981-1985, (Generation, Y). In all cases, the wage-supply response to belonging to small generations (early Millennial) is a high annual wage.

Secondly, Fig.2a shows how Low fertility permanently from 2050 creates a scarcity of workers 30-34 years old, resulting in higher wages $w_{30-34}$ from 2050. Higher fertility does not help wages of younger members ($L_{30-34}$) of Labor Supply, $L(t)$; Fig. (2b-2c), 2050-2090, show clearly how Higher fertility raise wages of middle-aged and older workers.

16 Belonging to (Y), $N_{30-34}$, t=2020, in footnote 10.
17 Belonging to (Y), $N_{45-49}$, t=2030, in footnote 10.
Fig. 2a. Annual wages for younger workers $w_{30-34}(t)$ in calendar years (t): 2020, 2090 - in three variants: Medium, Low, High fertility.

Source: The six numbers of $w_{30-34}(t)$ - for each variant - are seen in Table 5c.

Fig. 2b. Annual wages for the age group $w_{45-49}(t)$ in calendar years (t): 2020, 2090 - in three variants: Medium, Low, High fertility.

Source: The six numbers of $w_{45-49}(t)$ - for each variant - are seen in Table 5c.

Fig. 2c. Annual wages for older workers $w_{55-59}(t)$ in calendar years (t): 2020, 2090 - in three variants: Medium, Low, High fertility.

Source: The six numbers of $w_{55-59}(t)$ - for each variant - are seen in Table 5c.
Fig. 2d. Age-wage profiles, \( w_i(t) \), \( i = 1, 2, \ldots, M \), age group : \( i = 1 = 15-19, i = 11 = 65-69 \) - for the calendar years, \( t = 2010, 2020, 2030 \).

![Age-wage profiles](image)

Source: Sizes of annual wages, \( w_i(t) \), (41), (29), \( t = 2010, 2020, 2030 \), see Table 5a.

Fig. 2e. Age-wage profiles, \( w_i(t) \), \( i = 1, 2, \ldots, M \), \( i = 1 = 15-19, i = 11 = 65-69 \) - for the calendar years, \( t = 2010, 2040, 2050, 2070, 2090 \) : Medium fertility.

![Age-wage profiles](image)

Source: Annual wages, \( w_i(t) \), (41), (29), \( t = 2010, 2040, 2050, 2070, 2090 \), see Table 5b.

Whereas Fig. (2a-2c) focused on the fertility variants and showing their wage impacts upon particular age groups at selected time points, we exhibit in Fig. (2d) annual wages \( w_i(t) \) of all (overlapping) 11 age groups \( i \) - Age wage profile - for 3 calendar years \( t \). The observed Age wage profile of 2010, \( w_i(2010) \), Table 2 (col.6), is seen (black) in Fig. (2d) - and in Fig. (2e) (two panels), where \( w_i(t) \) is shown for 5 calendar years \( t \).

Apart from the last two age groups (60-64, 65-69), the shape of the calendar year \( t \) Danish Age-wage profiles, Fig. (2d, 2e), are qualitatively the same (concave) - a shape of Age-wage profiles that we shall later see extended to Cohorts \( T \) in Fig. (12, 13).
4.2 Age groups, LFP, Support ratios, Annual wages: 2020-2090

Table 6 provides demographic summary variables and wage incomes of national accounts. Row 1 gives the total working age population size, $N_{15-69}$, which corresponds to the Totals of $N_i$ in Tables 5a, 5b, (column 2). Row 2-4 give the population sizes, $N_{0-14}$, $N_{70+}$, $N$, from which the Danish Dependency ratios, (1) in Table 1b were derived.

Row 5 gives the Labor Force (Labor Supply), $L_{15-69}$, the Totals of $L_i$ in Tables 5a, 5b, (column 3), where the $L_i$ were generated by (39). Row 6 (ratio of row 5/row 1) give sizes of the macro (endogenous) Labor Force Participation rate ($l_{15-69}$), (LFP), (2).

The Danish macro LFP ($l_{15-69}$), (2), (39), for population projection period 2010-2090 are shown in Fig. 3 below. In the High fertility scenario, the LFP ($l_{15-69}$) is close to stationary in the 50 years from 2040 to 2090 as the population $N_{15-69}$ grows at the same rate as $L_{15-69}$. In the Low fertility scenario, we find some changes from 2030 to 2050, as the Population 15-69 years old, ($N_{15-69}$), falls more than the Labor Supply, ($L_{15-69}$), while both magnitudes fall at the same rate from 2050.

Fig. 4 illustrates (based on Table 6), the dramatic long-run consequences regarding the composition of the population by age groups outside the labor force. For the 0-14 years old ($N_{0-14}$), the range is between 10 and 20 percent of the population ($N$) for the Low, respectively the High fertility case. An even bigger range is found for the share of the population 70 years and older, $N_{70+}$. Until 2070, the upper part of Fig. 4 shows a low fertility 'dividend'. The shift in the last 20 years (2070-2090) is due to large increase in the dependency rate for the 70+ group, $d_o$, (1), cf. Fig. 1, in the Low fertility projection.

Row 7 gives the Support ratios, $L/N$, (3-5), for the period 2020-2090. The Support ratio (3) for 2010, Table 3a, was given in (6). While Support ratios for Denmark are widely available (World Bank and OECD, for example\textsuperscript{18}), our calculations in Tables (5a, 5b) show how $L = L_{15-69}$ in the Support ratio, Table 6 (row 5), are obtained as the Total of the same 11 age-specific Labor supplies $L_i(t)$ that are used for calculating the CRESH relative wages, (40), CRESH absolute wages, (41), seen in Tables (5a, 5b).

Danish Support ratios for the whole population, $L/N$, (3), (6), for 2010-2090 are shown in Fig. 5. Compared with the LFP ($l_{15-69}$) in Fig. 3, the only difference is - as

expected - found in the terminal year 2090, where the Support ratio (L/N) in the High fertility case is higher than in the Low fertility case, as a consequence of the increasing share of 0-14 years old, N_{0-14}/N, cf. Fig. 4.

Row 8 gives Total Wage Income (wL), as sum of all the 11 age groups \([L_i(t) \cdot w_i(t)]\) in Tables (5a, 5b), (columns 3, 5), in calendar year \((t)\).

Row 9 gives (as explained for Table 3a) similarly, Total Wage Income per capita, \(wL/N\), decomposed as : \(W_A \cdot \text{Support ratio}\), with \(W_A = w(2010) = 437552\). The \(w \cdot L(t)/N(t)\) for 2020-2090 are shown in Fig. 6. Over 30 years, 2020-2050, wage income per capita in Fig. 6 is significantly lower - the higher the fertility is - as high growth first in the younger parts \((L_i)\) of the Labor supplies, \(L(t)\), due to imperfect substitution with their CRESH substitution/complementarity parameters, \((\rho_i)\), Table 4, (Col.3) - implies lower productivity/wages. Over the next decades (after 2050) this effect is stabilised (stopped), as Higher fertility results in larger increases in the Labor supplies at all ages \((i)\). Moreover, at the Macro (aggregate) level we note the simple proportionality relation - \(W_A \cdot \text{Support ratio}\) - between Fig. 5 and Fig. 6.

The average annual wage \(W_A\) for our Labour aggregator (Aggregated Labour supplies), \(L_A(t) = L(t)\), is an exogenous constant for any Calendar year \((t)\) by assumption:

\[
\]

Aggregate wages \(wL\) is any year allocated to workers by age according to (12-15),(25-29). Despite (42) and Fig. (5,6), annual wages, \(w_i(t)\), of particular Age groups \((i)\) or Generations are certainly non-constant for Calendar years, as seen in Fig. (2a-2e). More on this below; cf. Table 8D, and Age group wages, \(w_i(t)\), as "shadow values" (marginal value-added : \(W_A \partial f/\partial L_i\)) in (84).

Row 10 provides the macro values of \((Y/N)\), (6), in accordance with Support ratio, \((L/N)\) (row 7), and macro Labour productivity, \(Y/L(2010) = 581972\) DKK, Table 3 (row 11). We have not shown \((Y/N)\) graphically over time - being just proportional to Fig. 5 with the constant, \(Y/L(2010)\). Thus, e.g., 2030, \((Y/N)\) is 213128 DKK in the Low variant, and 205629 DKK in the Medium variant, cf. Table 6 (row 10).
Table 6. Population Age Groups, Labor Supply, LFP and Support Ratios, Incomes per capita

<table>
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<th>Low Variant</th>
<th>High Variant</th>
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<td>3981520</td>
<td>3981520</td>
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<td>2124861</td>
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<td>929.739</td>
<td>929.739</td>
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Table 2, 2010: \(w = 473552 \text{ DKK}, \ L = 211472 \text{ Labor years}, \ wL = 924.317 \text{ Bill. DKK}, \ wL/N = 166615 \text{ DKK}, \ L/N = 0.3806 \text{ Y/L} = 581972 \text{ DKK}, \ Y/N = (Y/L)(L/N) = 221499 \text{ DKK}.

Source: Rows 1-4: United Nations (2015), cf. Tables (1,2); Rows 5-9: Tables (5a,5b).
Fig. 3. Labor Force Participation rates: Medium, Low, High fertility, 2010-2090.

Source: Danish \( L_{15-69}/N_{15-69} = l_{15-69}(t) \), (2), (4), (39), Tables (6, 3a, 2).

Fig. 4. Shares, \( n_{0-14}/N \), \( n_{70+}/N \), for Low, High fertility scenario, 2020-2090.

Source: Danish \( n_{0-14} \), \( n_{70+} \), (1), Tables (1, 6). \( \text{sum} \equiv n_{0-14} + n_{70+} = 1 - n_{15-69} = \frac{d}{1+d} \).

Fig. 5. Danish Support ratios - \( L/N \) -

Source: Danish \( L(t)/N(t) \), (3-6), Tables (6,3a).

Fig. 6. Wage Income per capita, 2020-2090.

Source: Danish \( W_A(2010) \cdot L(t)/N(t) \), Table 6.
5 Projected lifetime wage incomes of special cohorts

From the calendar (time, t) annual wages, \( w_i(t) \), of labor age groups, \( L_i(t) \), in Tables (5a,5b), we can for a particular cohort \( T \), extract the cohort annual wages, \( w_i^*(T) \), at each life-cycle age, \( i = 1, 2, \ldots, M = 11 \), where, age \( i = 1 = 15-19 \), age \( i = 2 = 20-24 \), etc.

The Labor supply (Labor inputs), \( L_i^*(T) \), of cohort \( T \) at life-cycle age \( (i) \) is related to the calendar Labor supplies, \( L_i(t) \) of age group \( i \), Tables (5a,5b), as follows:

\[
L_i^*(T) = L_i(T + [i - 1] 5) = L_i(t), \quad L_1^*(T) = L_1(t); \quad L_2^*(2020) = L_3(2030) = L_{25-29}(2030)
\]

Similarly for \( w_i^*(T) \).\(^{19}\) Cohort \( T \) annual wage age \( (i) \) and Calendar annual wage \( w_i(t) \):

\[
w_i^*(T) = w_i(T + [i - 1] 5) \equiv w_i(t), \quad i = 1, . . . , 11; \quad w_i^*(T) = w_1(t), \quad w_i^*(2020) = w_3(2030)
\]

All the results for, \( L_i^*(T), w_i^*(T) \), for every Cohort \( T \) are collected in Table 7.

\(^{19}\)By Tables (5a,5b) and (43), a few examples of extracting, \( w_i^*(T), L_i^*(T) \), Table 7, are:

Table 7. Medium Variant : Example - Cohort wages, \( w_i^*(T) \), of Cohort, \( T = 2010 \),

Table 5a: \( w_{15-19}^*(2010) = w_{15-19}(2010) = 187005; \quad w_{25-29}^*(2010) = w_{25-29}(2020) = 333172 \)

Table 5b: \( w_{45-49}^*(2010) = w_{45-49}(2040) = 467867; \quad w_{55-59}^*(2010) = w_{55-59}(2050) = 444392 \), Table 7.

Medium Variant : Example - Cohort wages, \( w_i^*(T) \), of Cohort, \( T = 2015 \),

Table 5a: \( w_{15-19}^*(2015) = w_{15-19}(2015) = 187053; \quad w_{30-34}^*(2015) = w_{30-34}(2030) = 393930 \)

Table 5b: \( w_{40-44}^*(2015) = w_{40-44}(2040) = 475811; \quad w_{50-54}^*(2015) = w_{50-54}(2050) = 459807 \), Table 7.

Table 7. Medium Variant : Example - Cohort wages, \( w_i^*(T) \), of Cohort, \( T = 2020 \),

Table 5a: \( w_{15-19}^*(2020) = w_{15-19}(2020) = 188933; \quad w_{25-29}^*(2020) = w_{25-29}(2030) = 340041 \)

Table 5b: \( w_{35-39}^*(2020) = w_{35-39}(2040) = 454562; \quad w_{65-69}^*(2020) = w_{65-69}(2070) = 468895 \), Table 7.

Table 7. High Variant : Example - Cohort wages, \( w_i^*(T) \), of Cohort, \( T = 2030 \),

Table 5a: \( w_{15-19}^*(2030) = w_{15-19}(2030) = 191356 \);

Table 5b: \( w_{35-39}^*(2030) = w_{35-39}(2050) = 484008; \quad w_{55-59}^*(2030) = w_{55-59}(2070) = 493656 \), Table 7.

Table 7. Low Variant : Example - Cohort wages, \( w_i^*(T) \), of Cohort, \( T = 2035 \),

Table 5b: \( w_{20-24}^*(2035) = w_{20-24}(2040) = 279471; \quad w_{50-54}^*(2035) = w_{50-54}(2070) = 459807 \), Table 7.

Thus Medium variant size of \( w_i^*(2010) \) for the Cohort 2010 at age 25-29 is 333172 (Table 7, row 6 column 2). This number was seen as \( w_i(2020) \) for 25-29 year old in 2020 (Table 5a, column 5, row 6).

For the Cohort 2020, the sizes of \( w_{15-19}(2020), w_{25-29}(2020), w_{35-39}(2020), w_{45-49}(2020) \) in Table 7 are the sizes seen in Tables 5a, 5b, (column 5) for, \( w_{15-19}(2020), w_{25-29}(2030), w_{35-39}(2040), w_{45-49}(2050) \), respectively.
The cohort annual wages, \( w^*_i(T) \), and cohort labor supplies, \( L^*_i(T) \), in Table 7 can be summed to generate the Total Life Wage Income of Cohort \( T \): \( w^*(T)L^*(T) \), where Labor Supply, \( L^*(T) \), is the Total Life Time Labor Supply of Cohort \( T \), and \( w^*(T) \) is the Average (Life) Annual Wage of Cohort \( T \), i.e., as defined in accordance with:

\[
\begin{align*}
\sum_{i=1}^{M} w^*_i(T)L^*_i(T) & = \sum_{i=1}^{M} L^*_i(T) ; \quad w^*(T) = \frac{\sum_{i=1}^{M} w^*_i(T)L^*_i(T)}{L^*(T)} \quad (44)
\end{align*}
\]

These Longitudinal (Cohort) Labor supplies, \( L^*_i(T) \), [life-cycle ages \( (i) \)], Longitudinal annual wages, \( w^*_i(T) \), and Life Time Cohort Labor supply, \( L^*(T) \), making the Average Annual Wage, \( w^*(T) \), are shown in three demographic Variants for six Labor Cohorts, \( T \), in Table 7 - where, \( T - 15 = t \), is Birth year \( (t) \) of the youngest Generation \( (t) \), which enter the Labor Cohort \( T \). Thus Cohort \( T=2035 \) (born 2020) (45) starts working \( t= 2035 \) and is retired in year, \( t = 2090 \) - the end year of Table 5b.

We gave above a few examples on how to extract (translate) information from Tables (5a,5b) to Cohorts in Table 7. Similarly many other numbers, \( w^*_i(T) \), \( L^*_i(T) \), in Table 7 can be traced back to Tables (5a,5b). However, for every Cohort \( T \), (45) - all longitudinally Cohort variables of \( w^*_i(T) \), \( L^*_i(T) \), exhibited in Table 7, contain much more information (numbers) than available in Tables (5a,5b), as evidently many needed intermittent calendar years (2015, 2025, 2035, 2045, etc.) are not shown in Tables (5a,5b). Hence the entire Table 7 has been obtained by completing all the additional calculations needed (but not shown) to extend the Tables (5a,5b).

It is important to fully realize, however, that it is all 11 overlapping age-group (cross-section) calendar wages, \( w_i(t) \), of many Calendar years \( (t) \) that generate - by equation, (43) - the relevant longitudinal annual wages, \( w^*_i(T) \), \( i= 1, ,11 \), of Labor Cohort \( T \) (and their Generation) through a working life-cycle of 55 years in 11 age-groups \( (i) \).

\[20\] The Average lifetime Wage (Earnings) of Cohort \( T \), \( w^*(T) \), (44), differs from Lifetime Earnings of a Cohort worker, optimally accumulating human capital (education, experience) and rentals (wages) during fixed lengths of working-life (ages/years). On shape of the life-cycle (Age-wage profiles) of such Cohort, see Rosen (1972, p.330; 1976, p.52), Welch (1979, p.79), and Berger (1984, p.590; 1985, p.572).
Table 7: Age-group Wages and Lifetime Wage Incomes of Cohorts, entering Labor market at age 15, size N15-19, in year 2010, ... 2030; Three Variants: Denmark

<table>
<thead>
<tr>
<th>Medium Cohort</th>
<th>2010 (N=151999)</th>
<th>2015 (N=155190)</th>
<th>2020 (N=158393)</th>
<th>2025 (N=161597)</th>
<th>2030 (N=164802)</th>
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<tr>
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<td>32.200</td>
<td>32.200</td>
<td>32.200</td>
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</table>

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<th>Low Cohort</th>
<th>2010 (N=151999)</th>
<th>2015 (N=155190)</th>
<th>2020 (N=158393)</th>
<th>2025 (N=161597)</th>
<th>2030 (N=164802)</th>
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<th>2020 (N=158393)</th>
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<td>924.317</td>
<td>924.317</td>
<td>924.317</td>
<td>924.317</td>
</tr>
</tbody>
</table>

Source: United Nations (2015) New York. Calculations based on the calendar years (1): 1991-1995 (Generation 1); Then: \(L_{24}\) refer to 2010-2014; Then: \(L_{24}\) refer to 2015-2019; Then: \(L_{24}\) refer to 2020-2024; Then: \(L_{24}\) refer to 2025-2029; Then: \(L_{24}\) refer to 2030-2034; Then: \(L_{24}\) refer to 2035-2039. For \(T = 2010\): \(L_{19}\) born calendar years (i): 1996-2000 (Generation 2); Then: \(L_{24}\) refer to 2010-2014; Then: \(L_{24}\) refer to 2015-2019; Then: \(L_{24}\) refer to 2020-2024; Then: \(L_{24}\) refer to 2025-2029; Then: \(L_{24}\) refer to 2030-2034; Then: \(L_{24}\) refer to 2035-2039. For \(T = 2030\): \(L_{19}\) born calendar years (i): 2011-2015. Then \(L_{24}\) refer to 2030-2034; Then: \(L_{24}\) refer to 2035-2039; Then: \(L_{24}\) refer to 2040-2044; Then: \(L_{24}\) refer to 2045-2049; Then: \(L_{24}\) refer to 2050-2054; Then: \(L_{24}\) refer to 2055-2059; Then: \(L_{24}\) refer to 2060-2064; Then: \(L_{24}\) refer to 2065-2069; Then: \(L_{24}\) refer to 2070-2074; Then: \(L_{24}\) refer to 2075-2079; Then: \(L_{24}\) refer to 2080-2084. For \(T = 2050\): \(L_{19}\) born calendar years (i): 2021-2025. Then \(L_{24}\) refer to 2021-2025; Then: \(L_{24}\) refer to 2026-2030; Then: \(L_{24}\) refer to 2031-2035; Then: \(L_{24}\) refer to 2036-2040; Then: \(L_{24}\) refer to 2041-2045; Then: \(L_{24}\) refer to 2046-2050; Then: \(L_{24}\) refer to 2051-2055; Then: \(L_{24}\) refer to 2056-2060; Then: \(L_{24}\) refer to 2061-2065. For \(T = 2080\): \(L_{19}\) born calendar years (i): 2031-2035. Then \(L_{24}\) refer to 2031-2035; Then: \(L_{24}\) refer to 2036-2040; Then: \(L_{24}\) refer to 2041-2045; Then: \(L_{24}\) refer to 2046-2050; Then: \(L_{24}\) refer to 2051-2055; Then: \(L_{24}\) refer to 2056-2060; Then: \(L_{24}\) refer to 2061-2065.
Fig. 7.
Annual wages of age group 35-39, $w^*_{{35-39}}(T)$, for 6 Cohorts 2010-2035
Medium, Low, High fertility

Source: $w^*_i(T)$, (43), $i = 35 - 39$, T, (45), in Table 7.

Fig. 8.
Annual wages of age group 55-59, $w^*_{{55-59}}(T)$, for 6 Cohorts 2010-2035
Medium, Low, High fertility

Source: $w^*_i(T)$, (43), $i = 55 - 59$, T, (45), in Table 7.

Fig. 9.
Ratio of the Annual Wages between stages (old/young) for 6 Cohorts T, 2010-2035
Medium, Low, High fertility.

Source: $w^*_{{55-59}}(T)/w^*_{{30-34}}(T)$ in Table 7, as obtained from: $w^*_i(T)$ in Fig. 7 - 8.

Fig. 10.
Average (annual wage) Life - all ages (i) - Income, $w^*(T)$, for 6 Cohorts T, 2010-2035
Medium, Low, High fertility.

Source: $w^*(T)$, (44), (45), in Table 7.
The sizes of the Generations/Cohorts are seen in Table 7 (second row) as, \(N_{15-19}(T)\), Table (5a,5b), i.e., specific size of Population age group (15-19) - youngest (15) entering Labor market, \(L_{15-19}(T)\) - in the years, \(T = 2010, 2015, 2020, 2025, 2030, 2035\).

In the Medium variant for example, \(N_{15-19}(2020) = 338740\) people aged 15-19 in 2020, which is also seen in Table 5a (row 1, column 2). For the same variant, \(N_{15-19}(2030) = 309940\) can be seen in Table 5a, (row 13, column 2). The \(N_{15-19}(2035) = 322890\) (not shown in Tables 5,6) will be 80-85 years of age in 2100, which is the last year of the United Nations population projections. The corresponding rows with \(N_{15-19}(T)\) for the Low and High variants are given further down in Table 7.

Generation sizes \(N_{15-19}(T)\) - and Cohort Lifetime Labor supply, \(L^*(T)\), (44) - are equivalent for all variants in all years, except for the last, Cohort (2035), since the fertility change commences in 2020 and takes until 2035 to be reflected in Cohort Labor, \(L^*(T)\).

Figures (7, 8) show Cohort \((T)\) annual wages for workers in the second half of the 30s, \(w_{35-39}^*(T)\), and for workers in the second half of the 50s, \(w_{55-59}^*(T)\). For the youngest age group, Fig. 7, significant changes are found when comparing the 2030 and the 2035 Cohorts. Here we find a strong impact in the High fertility case, where relative increase in younger workers, \(L_{35-39}^*(T)\), has a depressing effect on \(w_{35-39}^*(T)\). The counterpart to this is shown clearly in Fig. 8, where High fertility improves the position of older workers, \(w_{55-59}^*(T)\), for all 6 Cohorts, more so for the 2030 and 2035 Cohorts.

Fig. 9 presents an alternative illustration of how the ratio, old/young annual wages are affected for 6 Cohorts in three Fertility scenarios. Not surprisingly, old workers, \(w_{55-59}^*(T)\), are much better off relatively in the High compared to the Low fertility case.

Fig. 10 shows Cohort Average (Life-time) annual wage, \(w^*(T)\), for Cohorts T, 2010 to 2035, covering their full working life, summing up their wages at different ages (life cycle) in Fig. 7-8. Thus Cohort 2035 consists of workers 15-19 years \(L_{15-19}^*(2035)\) in year 2035, and of workers \(L_{65-69}^*(2035)\) retiring during 2086-2090, and living as 70-74 years old, \(N_{70+}(2015)\), in 2090. Even though the Cohort 2035 had lowest \(w_{35-39}^*(2035)\) with High fertility, then much better wages later as e.g., \(w_{55-59}^*(2035)\), ensured that the Average (Life time) wage, \(w^*(2035)\), were highest with High fertility. The importance for any Cohort \(w^*(T)\) of having many and large surrounding (cooperating) cohorts as
co-workers for the particular Cohort T (Generation) during its full working life (period).

The explicit wage formula of \( w^*(T) \), (44), with the analytic CRESH forms (40-41) of wage complementarity emphasize such interaction (mutual interdependence) behind \( w^*(T) \).

Fig. 10 compared future prospects of Cohorts using the Average Lifetime annual wages of the Cohort, \( w^*(T) \), shown in Table 7 (Row - Lifetime - in each variant). The value of \( w^*(T) \) is highest for the smallest Cohort, (2030), in Medium, High variants. For the Medium variant: \( w^*(2010)=422715, w^*(2015)=428173, w^*(2020)=436983, w^*(2025)=434895, w^*(2030)=451446, \) - depicted in Fig. 10 (blue). Thus Medium it Cohort (2030) has \( w^*(2030) \) as 5.2 percent higher\(^{21}\) than \( w^*(2015) \).

For Low Fertility variant, the highest is also seen for the smallest Cohort (T=1935). The relationships between Cohort size - measured by Cohort Labor supply, \( L^*(T) \) - and Average (life-time) annual wage, \( w^*(T) \), are illustrated for the three fertility scenarios in Fig. 11. The slopes are negative with Cohort Average (life-time) wages increasing with decreasing Cohort size. The exceptions are found for the High fertility cases where Average (life-time) annual wage is significantly higher for large 2025 and 2035 Cohorts, reflecting the positive wage impacts increased Labor supply of co-workers in other cohorts.

Indeed Cohort differences in \( w^*(T) \), Fig. 10-11, for all three demographic variants, are - with same parametric CRESH model for \( w_i^*(T) \), Fig. 7-9 - fully explained by wage complementarity differences that Cohort Labor supplies, \( L^*(T),^{22} \) (44) are exposed to.

In section 2.2, we saw in Table 2 (col.2) some positive/negative Population echo's of the sizes of earlier generations, and in section 4.1, we saw for calendar years in Table 5a, Fig. (2a-2c), the annual wage, \( w_i(t) \), effects of belonging to the small (early) Millennial generations, (1981-1985), (1986-1990). We have not shown (calculated) the Life time wage income, \( w^*(T) \), of first two Millennial generations (Cohorts, T=2000, T=2005), but they should be high as \( w^*(2000), w^*(2005) \) - not shown in Fig. 10-11. But \( w^*(T) \) of the last Millennial generation and the first Z - generation (Cohorts, T=2010, T=2015) are as \( w^*(2010), w^*(2015) \) in Fig. 10-11 - being lower than \( w^*(T) \) of more future Cohorts.

\(^{21}\)The \( N_{15-19}(2030) \), is is 13.5 percent smaller than \( N_{15-19}(2015) \), cf. Table 7.

\(^{22}\)The size of these \( w^*(T) \) effects depends on the degree of labor substitutability reflected in the parameter \( \rho_i \). If all \( \rho_i = 1 \), there is perfect labor substitution and sizes of Cohort/Labor supplies have no effect on its own relative (absolute) wages nor affect the relative (absolute) wage of other Cohorts.
Fig. 11. Life Time Cohort Labor Supply, $L^*(T)$, ("Cohort size"), and Cohort Average Life Income ($Annual wage$), $w^*(T)$ - for the six Cohorts, 2010-2035, in three variants: Medium, Low, High fertility.

![Cohort size and average cohort life wage income](image)

Source: Six numbers of $L^*(T)$, and, $w^*(T)$, (44), (45), seen (bottom) in Table 7.

Fig. 12. Annual wages - Age wage profile - for Cohort 2035, $w^*_i(T)$, $i = 1, 2, 11$, Cohort $T = 2035$, Generation, $t = 2020$,
- Medium, Low, High fertility.

![Age wage profiles - cohort 2035](image)

Source: $w^*_i(T)$, (43), $i = 1, 2, 11 \equiv i = 15-19, 20-24, 65-69$,
$T = 2035$, $t = 2020$, (45), from Table 7, (last Cohort, RHS).
We may trace some echo of small first Millennial Generation (1981-1985), Cohort (T=2005), on Life time wage income \( w^* \) of their descendants (progeny). Generation (1981-1985) is not exclusively - but it is the main Progenitor of Generation (2011-2015), Cohort T = 2030, and we do see an echo of first Millennial Cohort (T=2005) in Progenitor Cohort (T=2030) - as reflected in \( w^*(2030) \) - which indeed is the highest \( w^* \) in Table 7, Fig. 10-11, with smallest sizes of \( N_{15-19}(2030) = 309940, \) or \( L^*(2030) = 2.007.228. \)

Fig. 12 shows the longitudinal annual wages, \( w_i^*(T) \), to all ages (i) (life cycle) of the Cohort, T = 2035, for three fertility scenarios. Annual wages peak at ages 40-44, independently of the fertility scenarios. After this age, 40-44, differences in fertility has a clear impact with higher annual wages for older workers in the high fertility case, reflecting the scarcity of the older workers together with ample supplies of younger workers. The shape of the Age profile of annual wages, \( w_i^*(T) \), Fig. 12, applies qualitatively to any Cohort \( T \) in Table 7 (all vertical wage columns of \( w_i^*(T) \), to the left of Cohort 2035).

In Fig. 13, we show the Age-wage profiles, \( w_i^*(T) \), for \( T = 2010, 2020, 2030, \) Medium fertility - already seen with their Life time, \( w^*(T) \), for \( T = 2010, 2020, 2030 \) (on blue line) in Fig. 10. Hence Fig. 13 demonstrate that e.g., that the largest Life time, \( w^*(2030) \), in Fig. 10 also have the largest \( w_i^*(T) \) at any stage (all ages), (i), during entire working life (15-69).

We saw in Fig. (2d) that the smallest Generation (1981-1985) had - as the age group (45-49) in calendar year 2030 - the highest wages (above normal). Such above normal wages in calendar year 2010 is not just a temporary effect - but become a permanent effect - of being a small generation as (1981-1985). Thus such permanent wage effect of the small Generation (2011-15), Cohort, T = 2030, is seen for all ages (i) in Fig. 13.

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23The age composition of the labor force varies much over time due to demographic changes. The large post-war Generations, born 1946-1964 (defining American "baby boomers"), included four 5-year age groups, from the leading edge group (1946-50), peak in (1956-60), to trailing edge group (1961+), cf. Freeman (1979, p.289, Easterlin et al. (1990, p.281). Danish "baby boomers" refer to decade (1941-50).

The economic effects of several large "baby boom generations" (age-groups) are explored extensively in economic/demographic literature. "Twist [shift/rotation] in male age-wage profiles in late 1960s and early 1970s" (relative low earnings of younger workers) have empirically been attributed, Freeman (1979, p.315), Easterlin (1978, p.401), as impacts of large "baby boom" generations (1946-60) - the opposite of small generations (Y, 1981-90) effects, calendar year "twists", Fig. 2d - or as life-time impacts, Fig. 13.
Figure 13. Age-wage profiles, \( w_i^*(T) \), \( i = 1, 2, \ldots, M \), age group : \( i = 1 = 15-19, i = 11 = 65-69 \) - for the Cohorts, \( T = 2010, 2020, 2030 \) - Medium fertility variant.

Source: Cohort wages, \( w_i^*(T) \), (43), \( T = 2010, 2020, 2030 \), Table 7, horizontal top.

The overall Age-wage profiles as Fig. (12,13), Fig. (2d,2e), hold generally for Cohorts and Calendar years, and the shape of such Age-wage profiles are determined by the CRESH parametric Labor Aggregator, (7-8), (12-14), (15), or obtained by the dual CRESH Age-Wage [Inverse Labor demand] \( w_i - form \), (25-29), (81), (84), (92-93).

6 Population, Division of Labor, and Wages

Demography, Population, Labor Allocation, Wages, and National Income per capita are subjects of classic fields and studies in Political Economy/Economics. Let us end with a few literature comments provided for both inductive and deductive aspects of this paper. For this purpose, it is useful to recall the macro relations and the ratios in (5-6), (25),

\[
\frac{Y}{N} = \frac{Y}{L} \cdot \frac{L}{N} ; \quad \frac{L}{N} = \sum_{i=1}^{11} l_i \cdot n_i = l_{15-69} \cdot n_{15-69} ; \quad \frac{W}{N} = \frac{W}{L} \cdot \frac{L}{N} = W_A \cdot \frac{L}{N} ; \quad W_A L_A = \sum_{i=1}^{M} w_i L_i \equiv W \quad (46)
\]

As to proportions in the Per Capita National Income ("Wealth of Nation") identity, (46), Smith (1790,1961, p.1) opens with the statement: "The annual labour of every nation is the fund, which supplies it with all the necessaries and conveniences of life. - According therefore, this produce [product, output, \( Y \)] bears a greater or smaller proportion [\( Y/N \)] to number [\( N \)] of those who are to consume it. But this proportion [\( Y/N \)] must in every nation be regulated by two different circumstances : 1. the skill, dexterity, and judgement

43
with which its labour is generally applied [Labor productivity, Y/L] 2. the proportion [L/N] between the number of those who are employed in useful labor [L] and those not so employed [N-L]. - The abundance or scantiness of this supply [per-capita produce, Y/N] seems to depend more upon the former [Y/L] of those two circumstances than upon the latter” [L/N] (italics ours).

The Employment/Population (“Support”) ratio, (L/N), (bounded above by one\textsuperscript{24}) is always much lesser than one as the numerical size of (L/N), (46), is by definition the product of \(LFP = l_{15-69}\) (interval: 0.5-0.6), and the Working population share, \(n_{15-69}\) (interval: 0.7-0.6), i.e., e.g., \(L/N = l_{15-69} \cdot n_{15-69} = 0.38\), cf. (5-6), and Tables (1,2,6).

As to (Y/L), Smith (1790, p.7) says: ”The greatest improvements in the productive powers of labour, and the greater part of the skill, dexterity, and judgement with which is anywhere directed, or applied, seems to have been the effects of the division of labour. - p.11: This great increase of the quantity [Output] of work [Labor productivity,Y/L] which, in consequence of the division of labour, the same number of people [L] are capable of performing, is owing to three different circumstances 1. the increase of dexterity in every particular workman 2. the saving of the time which is commonly lost in passing from one species of work to another 3. the invention of a great number of machines \([K, K]\) which facilitate and abridge labour, and enable one man to do the work of many” (italics ours).

The Productive powers of Labor by the Division of Labor (the notions used by Smith above), is economically and conceptually expressed with: Production functions, \(Y = F(L,K) = L \cdot f(K/L) \equiv L \cdot f(k)\), or as, \(F(Y, L_I, L_{II}, K_{III}, K_{IV}, K_{V}) = 0\), cf. (34).

Production (Division of Labor) by different qualities of workers according to skills [education, training], dexterity and judgement [age/maturity/experience] provide the framework for analyzing differences in factor prices - earnings structures, Smith (1790, p.111)\textsuperscript{25}: ”Pecuniary wages and profits [rentals of machinery], indeed, are everywhere in Europe extremely different according the different employments of labour and stock” (italics ours).

As to Pecuniary wage distributions for the complete set of demographic age-groups of the National Labor supply - making Total Wages (W), (46), in the National income (Y),

\textsuperscript{24}[Modern Growth Theory and Macroeconomics, cf. standard exposition, Solow (2000), Romer (2019), do not allow in any of the models for the distinctions between Labor (L) and Population (N).

\textsuperscript{25}As to the wage structure analysis in Smith (1790, Ch. 10), see Katz and Autor (1999, p.1464).
Table 3, our CRESH Labor Aggregator, $L_A = f(L_1, L_2, \ldots, L_M)$ - and wage generator by its derivatives, (27-29) - formed the annual wage distributions: Calendar year Age-wage profiles in Tables 5a-5b - and implied the Cohort Life time wages in Table 7.

It must be emphasized that the calculated results in Tables (5a-5b, 7) are not dealing with a pure Labor Economy, using only distinct Labor inputs. As stressed by Smith above, Output (Y) by 'Division of Labor' and Labor productivity, $Y_T = y$, involved 'machinery' [K] in Production functions, $Y = F(L, K)$, $y = f(k)$; let, $p \frac{\partial Y}{\partial k} = W_A(k)$:

$$w = p \frac{\partial Y}{\partial L} ; r = p \frac{\partial Y}{\partial K} ; w = \frac{w_Y}{\partial L} = p \cdot y - r \cdot k = W_A ; \quad k = 2.72 : W_A(2.72) = 582 - 0.053 \cdot 2.72 = 438 \quad (47)$$

where actual numbers of Table 3 are used in (47-48), RHS. Next, we have by (47), (29):

$$\lambda_i(t) = \frac{L_i(t)}{L(t)} = \frac{l_i \cdot n_i(t)}{l_{15-69}(t) \cdot n_{15-69}(t)} ; \quad l_i = l_i(2010) , \quad i = 1, 2, \ldots, M ; \quad M = 11$$

$$w_i(t) = \tilde{w}_4(t) \cdot \alpha_i \rho_i \lambda_i(t)^{\rho_i - 1} ; \quad k = 2.72 , \quad p \cdot y = 581972 , \quad w = W_A = 437552 \quad DKK \quad (48)$$

Thus, $w_i(t)$, $i = 1, 2, \ldots, M$, (48), give all pecuniary (money) wages in Tables (5a-5b), and/or as exhibited in any/every Figure 2-6. As seen in (47-48), $\forall t : k(t), y(t), W_A(t)$, are unchanged during projection period 2010-2090; $W_A(t)$ is the arithmetic mean wage rate of the nation’s year-round, full-time workers, (46) - exogenous, $W_A(2010)$, cf. (42).

Around such arithmetic mean, Macro wage rate, $W_A(2010)$, however, the 'Division of Labor' with different Ages (maturity/experience) of workers amply generate at Micro level a changing wage structure over time, given by the Age wage profiles, $w_i(t)$, $i = 1, 2, \ldots, M$, (48). The explicit form (48) shows that the money age-wage determinants are: 1. $[n_i(t), l_{15-69}(t), n_{15-69}(t)]$, by affecting continuously changing, $\lambda_i(t)$, Age distributions\(^{26}\) of demographic induced Labor supplies, $L(t) = \sum_{i=1}^{M} L_i(t)$, (39), due to changing Employment/Population (Support) ratio, $L(t)/N(t)$, 2. the endogenous money wage, $\tilde{w}_4(t)$, (29), 3. the CRESH parameters, $\alpha_i, \rho_i$ in Table 4.

The Age-wage solutions, $w_i(t)$, $i = 1, 2, \ldots, M$, (48) in Tables (5a-5b), Fig. 2-6, with changing Support ratio, $L(t)/N(t)$, may be considered as Micro Age-wage scenarios evolving under Macro 'steady-state' conditions ['steady-state' sizes of aggregate capital-labor ratio (k), aggregate labor productivity (y), aggregate wage, $w = W_A$].

\(^{26}\)Edin and Holmlund, (1995, p.328-29) show how marked fluctuations ('shocks') in calendar year sizes of Swedish (Totals, $N_{15-19}$) translate into substantial changes in Age distribution (ratios), $n_i(t)$, $i = 1, \ldots, M$ - coinciding with rising/falling Youth relative wages. Adjusting $n_i(t)$ changes $\lambda_i(t)$ in entire Age distribution of $L(t)$ - affecting wage structure/calendar year Age wage profiles, $w_i(t)$, $i = 1, \ldots, M$, (48).
The shape (qualitative properties) of the quantitative Age-wage profiles (48) will be robust and carry over to 'non-steady-state' conditions with increasing aggregate Labor productivity \((Y/L = y)\) and increasing per capita National Income, (46) - as the result of Capital Accumulation beyond 'capital widening' to 'capital deepening' [increasing capital-labor ratios, \(k(t)\)] in well-known macro-, two sector\textsuperscript{27}, and multisector growth models. Total (Aggregate) Labor supply, \(L(t)\), in such growth models could still be the demographic induced Labor supplies, \(L(t) = \sum_{i=1}^{M} L_i(t)\), (39), that could allow for also generating Micro age-wage profiles, \(w_i(t)\), \(i = 1, 2, .., M\), from quantitative growth models.

7 Final Comments and Conclusion

This paper generalized models of imperfect labor substitution/complementarity by simultaneously: (i) specifying the CRESH Labor Aggregator function - relaxing the assumption of single-level, Arrow et al. (1961, p.230), CES elasticity of substitution between labor age groups - dually CES complementarity elasticities of wages to age-group supplies, (ii) allowing for a much larger number of age groups than is common in the literature, (iii) CRESH modelling the evolution and consequences of several demographic variants over longer transition periods rather than having a constant age distribution of the population, the labor force, and within and between the cohorts.

We have quantitatively demonstrated the micro-macro economic impacts of the assumptions - alternative fertility scenarios in the demographic projections (2020-2090) - on calendar year (t) wage patterns (Age-wage profiles) in the 'short-run', coming years (decade), and in the 'long-run' upon the lifetime wage incomes for selected (Generations), Cohorts (T), within the period (2010-2035).

The CRESH Labor Aggregator functional form can easily be analytically extended (specified) to include CRESH Subaggregator functions for any relevant Disaggregated Labor categories. Furthermore, the CRESH Labor aggregate (or Labor subaggregates) can next be combined with other Production factors (Capital inputs) in proper specified CRESH Multi-factor Production functions to be applied in single-sector (Macro) or multi-

\textsuperscript{27}Equipment investment are among prime determinants to national growth performance (productivity, per capita growth), Jensen (2003, p.82). Machinery becomes "cheap as well as good," Mokyr (1990, p.87).
sector GE models. In such interaction, the National Aggregate wage, \( W_A \), become endogenously generated and can provide unified macro equilibrium feedback in calendar years to forming the Age-wage profiles of Labor age-groups and to selected Labor cohorts.

We have come a long way and reached a higher vantage point, which offer a better outlook and apprehension of the roads passed. In closing, we look forward to see Demography and in particular Labor Economics promoting coherent quantifications and projections of real-world (calendar) Annual wages and full-time Employment (Labor years), based on relevant Demographic Register data and consistent with National Income Accounts.
8  App.A : Labor Substitution and Complementarity

8.1 Substitution elasticities and complementarity elasticities

Allen-Uzawa partial substitution elasticities, \( \sigma_{ij} \) of any factor pair, \((L_i, L_j)\), for CRESH, (7-8) - Hanoch (1971, p.699), Hanoch (1979, p.296), Guest & Jensen (2016, p.29) - are:

\[
\sigma_{ij} = \frac{1}{(1-\rho_i)(1-\rho_j)} \bar{\rho} = \sigma_{ji} > 0 \ , \ i \neq j \ ; \ \bar{\rho} = \sum_{i=1}^{M} \frac{\varepsilon_i}{1-\rho_i} \quad (49)
\]

\[
\sigma_{ii} = \frac{1}{(1-\rho_i)} \left[ \frac{1}{(1-\rho_i)} \bar{\rho} - \frac{1}{\varepsilon_i} \right] < 0 \ , \ i = 1, \ldots, M \quad (50)
\]

where \((\sigma_{ii})\) are the “total substitution elasticity” terms; the variable \((\bar{\rho})\) is a weighted average of the parameters, \(1/(1-\rho_i)\), with the respective wage (cost) shares \((\varepsilon_i)\) as variable weights. Clearly, especially larger values of \(\rho_i\) and \(\rho_j\) give a larger \(\sigma_{ij}\). The restrictions (8) imply that \(\sigma_{ij} > 0\) : all CRESH labor inputs \(L_i\), \(i=1,\ldots,M\), are substitutes.

If all \(\rho_i > 0\), then all \(\sigma_{ij} > 1\), (49). Note also that any \(\sigma_{ij}\) given by (49) via shares \(\varepsilon_i\), (14), depends on all the parameters, \(\rho_i, \alpha_i\), and all the Labor inputs \(L_i\), \(i = 1,\ldots,M\).

It follows from (49) that, although all \((\sigma_{ij})\) are variable elasticities of substitution (VES), they have nevertheless an invariant (constant) CRESH pattern:

\[
\frac{\sigma_{ik}}{\sigma_{jk}} = \frac{(1-\rho_j)}{(1-\rho_i)} \ ; \ \rho_i > \rho_j \ : \ \sigma_{ik} > \sigma_{jk} \ ; \ \forall k \neq i, j \ ; \ \frac{\sigma_{ij}}{\sigma_{kl}} = \frac{(1-\rho_k)}{(1-\rho_i)(1-\rho_j)} \quad (51)
\]

The restrictions (8) and expressions (49-51) were obtained by Hanoch (1971, p.698) via Lagrangian cost minimizing factor demand functions that correspond to a unique CRESH minimum Cost function, \(C(\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_M, \mathbf{L}_A)\), or unit cost functions, \(c(\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_M)\),

\[
C(w_1, w_2, \ldots, w_M, L_A) = c(w_1, w_2, \ldots, w_M) L_A = \sum_{i=1}^{M} w_i L_i \ ; \ L_i = \frac{\partial C}{\partial w_i} ; \ E(C,w_i) = \varepsilon_i \quad (52)
\]

dual to the implicit CRESH production (aggregator) function, (7-8), (10), \(\varepsilon_i = \frac{w_i L_i}{C} \), (14).

The own-price/cross-price factor demand elasticities corresponding to (49-50), (52), are:

\[
E(L_i, w_i) = \varepsilon_i \sigma_{ii} \ ; \ E(L_i, w_j) = \varepsilon_j \sigma_{ij} \ ; \ E(L_j, w_i) = \varepsilon_i \sigma_{ji} \ ; \ i = 1, \ldots, M \quad (53)
\]

\(E(L_i, w_i)\), \(E(L_i, w_j)\) are conditional (compensated, fixed : \(L_A\)) Labor demand elasticities. Like two-factor production/cost functions, the changes in factor shares \((\varepsilon_i)\) are ruled by,

\[
\frac{\partial \varepsilon_i}{\partial w_j} \geq 0 \iff \sigma_{ij} \geq 1 \iff E(\varepsilon_i, w_j) = \varepsilon_j (\sigma_{ij} - 1) \ ; \ i \neq j \ , \ i = 1, \ldots, M \quad (54)
\]
The CRESH elasticities, (53) and (49-50), satisfy the standard summation properties:

\[ \sum_{j=1}^{M} E(L_i, w_j) = \sum_{j=1}^{M} \varepsilon_j \sigma_{ji} = 0; \quad \sum_{j=1}^{M} \varepsilon_j E(L_j, w_i) = 0 \tag{55} \]

Actual parametric CRESH substitution elasticities, \( \sigma_{ij} \), \( \sigma_{ii} \), (49-50), and factor demand elasticities, \( E(L_i, w_i) \), \( E(L_i, w_j) \), (53), (55), are shown in Table 8A-8B, using the validated CRESH parameter values: \( (\rho_i, \alpha_i) \) in Table 4, (column 3-4), and the Labor inputs \( L_i, i=1,\ldots,M \), \( L_A = L \), (2010), in Table 2 (column 5).

The \( \sigma_{ij} \) formulas (56) are the Uzawa (1962, p.293) duality forms of Allen (1938, p.504)

\[ \sigma_{ij} = c_i \frac{\partial^2 C}{\partial w_i \partial w_j}, \quad \sigma_{ii} = c_i \left( \frac{\partial^2 C}{\partial w_i \partial w_i} \right)^2; \quad c_{ij} = \frac{\partial^2 f}{\partial L_i \partial L_j}, \quad c_{ii} = \frac{\partial^2 f}{\partial L_i \partial L_i} \tag{56} \]

The partial elasticity of substitution (\( \sigma_{ij} \)). But it is impossible to apply the beautiful and simple \( \sigma_{ij} \) formulas (56) to get the CRESH results (49-50), as the relevant dual CRESH cost function, \( C(w_1, w_2, \ldots, w_M, L_A) \), (52), has no closed form. However, such existing unknown dual CRESH Cost function (52) would by \( \sigma_{ij} \) (56) give the same CRESH parametric substitution elasticities (49-50) - as were successfully derived from the first and second order conditions for CRESH Lagrangian cost minimization by Hanoch (1971, p.697-98).

The Hicks partial complementarity elasticity \( (c_{ij}) \), (56), for any factor pair \( (L_i, L_j) \) of CRESH function, \( L_A = f(L_1, L_2, \ldots, L_M) \), (10), are defined exactly in analogy with \( \sigma_{ij} \) of C, (52), see Sato and Koizumi (1973, p.47)

\[ f = f(L_1, L_2, \ldots, L_M) \]

Note that the size of \( L_A \) [level of output, \( Y \) (note 35)] is not held constant in complementarity elasticities, \( c_{ij} \). In fact, positive \( c_{ij} \) measures exactly the degree to which two factor inputs jointly contribute to a change in \( L_A [Y] \) - as the cross-partial derivative \( \frac{\partial^2 f}{\partial L_i \partial L_j} \) shows in (56).

Thus in contrast to \( \sigma_{ij} \), (49), larger values of \( \rho_i \) and \( \rho_j \) give smaller numbers for \( c_{ij} \), (57).

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28 Sato & Koizumi (1973, p.46) considered an explicit production function as, \( Y = F(X_1, X_2, \ldots, X_M) \), \( Y = output, X_i = i\text{-th input, with derivatives } \forall X_i > 0 : \frac{\partial F}{\partial X_i} > 0, \frac{\partial^2 F}{\partial^2 X_i} < 0, Y = \sum_{i=1}^{M} \frac{\partial F}{\partial X_i} X_i \).

The complementarity elasticities, \( c_{ij} \), \( c_{ii} \), are defined as:

\[ c_{ij} = \frac{F \frac{\partial^2 F}{\partial X_i \partial X_j}}{\frac{\partial F}{\partial X_i}}, \quad c_{ii} = \frac{F \frac{\partial^2 F}{\partial X_i \partial X_i}}{\frac{\partial F}{\partial X_i}} \]

The problem with applying these \( c_{ij} \), \( c_{ii} \), definitions to CRESH function, \( L_A = f(L_1, L_2, \ldots, L_M) \), above in (56) is that \( f \), (10), is not - as here \( F(X_1, X_2, \ldots, X_M) \) - an explicit function. However, we know (can calculate, as explained in section 8.2 and Appendix B) the derivatives of \( f \), (10), to use in (56), and \( L_A = f \) drops out of (56) - as seen from CRESH formulas (72-73), (78-79), and finally stated in (57-58).

Table 8A. Partial substitution elasticities ($\sigma_{ij}, \sigma_{ii}$) - Denmark 2010 - by (49-50) and Table 4.

<table>
<thead>
<tr>
<th></th>
<th>$L_1$</th>
<th>$L_2$</th>
<th>$L_3$</th>
<th>$L_4$</th>
<th>$L_5$</th>
<th>$L_6$</th>
<th>$L_7$</th>
<th>$L_8$</th>
<th>$L_9$</th>
<th>$L_{10}$</th>
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<td>5.786</td>
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<td>3.858</td>
<td>2.893</td>
<td>2.315</td>
<td>2.315</td>
<td>2.893</td>
<td>3.858</td>
<td>5.786</td>
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<td>5.786</td>
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<td>-26.883</td>
<td>2.893</td>
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<td>2.315</td>
<td>2.893</td>
<td>-23.6932</td>
<td>5.786</td>
<td>5.786</td>
</tr>
</tbody>
</table>

\[ \sum_{i=1}^{n} \epsilon_i \sigma_{ii} = 0.000 \]

Table 8B. Conditional factor demand elasticities, $E (L_i, w_i)$, $E (L_i, w_j)$ - by (53) and Table 4.

<table>
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<th>$w_4$</th>
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<th>$w_7$</th>
<th>$w_8$</th>
<th>$w_9$</th>
<th>$w_{10}$</th>
<th>$w_{11}$</th>
<th>$\sum_{i=1}^{n} E(L_i, w_i)$</th>
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<td>0.375</td>
<td>0.627</td>
<td>0.612</td>
<td>0.543</td>
<td>0.537</td>
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<td>0.571</td>
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<td>0.215</td>
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<td>0.228</td>
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<td>0.150</td>
<td>0.251</td>
<td>0.245</td>
<td>0.217</td>
<td>-1.785</td>
<td>0.238</td>
<td>0.280</td>
<td>0.228</td>
<td>0.041</td>
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<tr>
<td>$L_8$</td>
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<td>0.188</td>
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<td>0.271</td>
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</tr>
<tr>
<td>$L_9$</td>
<td>0.035</td>
<td>0.189</td>
<td>0.250</td>
<td>0.418</td>
<td>0.408</td>
<td>0.362</td>
<td>0.358</td>
<td>0.397</td>
<td>-2.867</td>
<td>0.381</td>
<td>0.068</td>
<td>0.000</td>
</tr>
<tr>
<td>$L_{10}$</td>
<td>0.053</td>
<td>0.284</td>
<td>0.375</td>
<td>0.627</td>
<td>0.612</td>
<td>0.543</td>
<td>0.537</td>
<td>0.596</td>
<td>0.700</td>
<td>-4.429</td>
<td>0.101</td>
<td>0.000</td>
</tr>
<tr>
<td>$L_{11}$</td>
<td>0.053</td>
<td>0.284</td>
<td>0.375</td>
<td>0.627</td>
<td>0.612</td>
<td>0.543</td>
<td>0.537</td>
<td>0.596</td>
<td>0.700</td>
<td>0.571</td>
<td>-4.899</td>
<td>0.000</td>
</tr>
</tbody>
</table>

\[ \sum_{i=1}^{n} \epsilon_i E(L_i, w_i) = 0.000 \]
8.2 Labor Complementarity elasticities

The elasticity $c_{ij}$ formulas (56) are simple; but the CRESH, $L_A = f(L_1, L_2, ..., L_M)$, (10), did not exist in closed form. For CRESH, (7-8), the complementarity elasticities $c_{ij}$, (56), are explicitly derived in Appendix B, (78-79), as parametrically given by:

$$c_{ij} = 1 - \rho_i - \rho_j + \hat{\rho} = c_{ji}, \ i \neq j; \ \hat{\rho} = \sum_{i=1}^{M} \varepsilon_i \rho_i$$ (57)

$$c_{ii} = 1 - 2\rho_i - (1 - \rho_i)/\varepsilon_i + \hat{\rho} < 0, \ i = 1, ..., M$$ (58)

where $(c_{ii})$ are the “total complementarity elasticity” terms; variable $(\hat{\rho})$ is a weighted average of parameters $(\rho_i)$, with the respective wage (cost) shares $(\varepsilon_i)$ as variable weights.

CRESH $(c_{ij})$ are all variable complementarity elasticities, (57), but they have an invariant (constant) CDEC (constant difference of elasticity of complementarity) pattern:

$$c_{ik} - c_{jk} = \rho_i - \rho_j; \ c_{ij} - c_{kl} = (\rho_i + \rho_j) - (\rho_k + \rho_l)$$ (59)

Note that unlike substitution elasticities, $\sigma_{ij}$, (49), the restrictions (8) do not impose a particular sign upon all the complementarity elasticities, $c_{ij}$, (57).

Wage Income function, $W = W(L_1, L_2, ..., L_M, W_A)$ - as a dual to Wage Cost function, $C(w_1, w_2, ..., w_M, L_A)$, (52) - is an alternative Wage Sum formulation with important applications in sections 4-5 for the elasticities $c_{ij}$, (57), see Appendix B, (81-88):

$$W(L_1, L_2, ..., L_M, W_A) = W_A f(L_1, L_2, ..., L_M) = W_A L_A ; \ w_i = \frac{\partial W}{\partial L_i}, \ E(W, L_i) = \varepsilon_i$$ (60)

$$W(L_1, L_2, ..., L_M, W_A) = c(w_1, w_2, ..., w_M) L_A = W_A L_A = \sum_{i=1}^{M} w_i L_i ; \ W_A = c(w_1, w_2, ..., w_M)$$ (61)

$$c_{ij} = c_{ij}(L_1, L_2, ..., L_M) = \frac{W}{\partial L_i \partial L_j} ; \ i = 1, ..., M, \ j = 1, ..., M; \ c_{ii} = c_{ii}(L_1, L_2, ..., L_M) = \frac{W}{\partial L_i \partial L_i}$$ (62)

Factor price (wage) elasticities w.r.t own-, cross supply increases are, cf. (53), (57-58),

$$E(w_i, L_i) = \varepsilon_i, \ c_{ii}, \ E(w_i, L_j) = \varepsilon_j c_{ij}, \ E(w_j, L_i) = \varepsilon_i c_{ji}; \ i = 1, ..., M.$$ (63)

$E(w_i, L_i), E(w_i, L_j)$, are conditional (fixed $W_A$) partial wage elasticities of group $(i)$. Like two-factor production/cost functions, wage shares $(\varepsilon_i)$, (14), (60), follow the rules:

$$\frac{\partial \varepsilon_i}{\partial L_j} > 0 \iff c_{ij} > 1 \iff E(\varepsilon_i, L_j) = \varepsilon_j (c_{ij} - 1); \ i \neq j, \ i = 1, ..., M$$ (64)
Note that \( c_{ij} \) in (57) depend on all parameters, \( \rho_i, \alpha_i \), and also via: \( \varepsilon_i, (\hat{\rho}) \), on all the Labor inputs, \( L_i, i = 1, \ldots, M \), cf. (14), (49). Thereby is \( c_{ij} \) the relevant and adequate tool (summary measure) - with our CRESH forms, (57) - to answer distributional (absolute wage share) issues with formula (64); cf. (54). See Sato and Koizumi (1973, p.486).

CRESH elasticities, (62-63), (57-58), (89), have standard summation properties, cf. (55):

\[
\sum_{j=1}^{M} E(w_i, L_j) = \sum_{j=1}^{M} \varepsilon_j c_{ji} = 0 \quad ; \quad \sum_{j=1}^{M} \varepsilon_j E(w_j, L_i) = 0
\]  
(65)

Actual parametric CRESH complementarity elasticities, \( c_{ij}, c_{ii}, (57-58) \), and the wage effect elasticities, \( E(w_i, L_i), E(w_i, L_j), \) (63), (65), are shown in Tables 8C-8D, cf. CRESH parameter values: \( (\rho_i, \alpha_i) \) in Table 4, (column 3-4), and the Labor inputs \( L_i, i=1,\ldots,M, L_A = L, \) (2010), in Table 2 (column 5).

Note in Table 8D that the numerically highest wage elasticities, \( E(w_i, L_i) \), are: \( E(w_5, L_5), E(w_6, L_6), E(w_7, L_7), E(w_8, L_8), \) i.e., being most sensitive to own supply increases. These same middle age groups gain most by larger cross supplies from other age-groups, i.e., have the highest wage elasticities, \( E(w_i, L_j), i = 5,6,7,8, i \neq j \), in Table 8D - they have also the largest cross complementarity elasticities in Table 8C.

Finally, let us note from (49) and (57), that if \( \forall \rho_i = \rho \), cf. CES, (9), then we have,

\[
\forall \rho_i = \rho : \quad \sigma_{ij} = \frac{1}{(1-\rho_i)(1-\rho_j)\hat{\rho}} = \frac{1}{1-\rho} ; \quad c_{ij} = 1 - \rho_i - \rho_j + \hat{\rho} = 1 - \rho
\]  
(66)

i.e., substitution elasticities (\( \sigma_{ij} \)) and dual complementarity elasticities (\( c_{ij} \)) are simply reciprocals of each other, and there would also be simple "reciprocal" relations between factor demand elasticities (53) and the so-called "inverse factor demand" [conditional partial wage] elasticities, (63). But with the much richer parametric class of CRESH production/aggregator functions and their duality relations, the simple reciprocals in (66) evidently no longer apply - and clearly Table 8C is neither the reciprocal of Table 8A.

With demographic Age groups and exogenous Labor supplies \( (L_i) \), (39), it is \( c_{ij}, (57-58) \), and \( E(w_i, L_i), E(w_i, L_j), E(w_j, L_i), E(w_i, L_j), \) (63-65), Tables 8C-8D that are the relevant elasticities - which are behind all the Age-wage group results in Tables 5-7.
Table 8C. Partial complementarity elasticities ($c_{ij}$, $c_{ii}$) - Denmark 2010 - by (57-58) and Table 4.

<table>
<thead>
<tr>
<th></th>
<th>$L_1$</th>
<th>$L_2$</th>
<th>$L_3$</th>
<th>$L_4$</th>
<th>$L_5$</th>
<th>$L_6$</th>
<th>$L_7$</th>
<th>$L_8$</th>
<th>$L_9$</th>
<th>$L_{10}$</th>
<th>$L_{11}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L_1$</td>
<td>-32.713</td>
<td>0.022</td>
<td>0.122</td>
<td>0.122</td>
<td>0.222</td>
<td>0.322</td>
<td>0.322</td>
<td>0.222</td>
<td>0.122</td>
<td>0.022</td>
<td>0.022</td>
</tr>
<tr>
<td>$L_2$</td>
<td>0.022</td>
<td>-6.089</td>
<td>0.122</td>
<td>0.122</td>
<td>0.222</td>
<td>0.322</td>
<td>0.322</td>
<td>0.222</td>
<td>0.122</td>
<td>0.022</td>
<td>0.022</td>
</tr>
<tr>
<td>$L_3$</td>
<td>0.122</td>
<td>0.122</td>
<td>-4.402</td>
<td>0.222</td>
<td>0.322</td>
<td>0.422</td>
<td>0.422</td>
<td>0.322</td>
<td>0.222</td>
<td>0.122</td>
<td>0.122</td>
</tr>
<tr>
<td>$L_4$</td>
<td>0.122</td>
<td>0.122</td>
<td>0.222</td>
<td>-2.545</td>
<td>0.322</td>
<td>0.422</td>
<td>0.422</td>
<td>0.322</td>
<td>0.222</td>
<td>0.122</td>
<td>0.122</td>
</tr>
<tr>
<td>$L_5$</td>
<td>0.222</td>
<td>0.222</td>
<td>0.322</td>
<td>0.322</td>
<td>-2.415</td>
<td>0.522</td>
<td>0.522</td>
<td>0.422</td>
<td>0.322</td>
<td>0.222</td>
<td>0.222</td>
</tr>
<tr>
<td>$L_6$</td>
<td>0.322</td>
<td>0.322</td>
<td>0.422</td>
<td>0.422</td>
<td>0.522</td>
<td>0.622</td>
<td>0.622</td>
<td>0.522</td>
<td>0.422</td>
<td>0.322</td>
<td>0.322</td>
</tr>
<tr>
<td>$L_7$</td>
<td>0.322</td>
<td>0.322</td>
<td>0.422</td>
<td>0.422</td>
<td>0.522</td>
<td>0.622</td>
<td>0.622</td>
<td>0.522</td>
<td>0.422</td>
<td>0.322</td>
<td>0.322</td>
</tr>
<tr>
<td>$L_8$</td>
<td>0.222</td>
<td>0.222</td>
<td>0.322</td>
<td>0.322</td>
<td>0.422</td>
<td>0.522</td>
<td>0.522</td>
<td>0.422</td>
<td>0.322</td>
<td>0.222</td>
<td>0.222</td>
</tr>
<tr>
<td>$L_9$</td>
<td>0.122</td>
<td>0.122</td>
<td>0.222</td>
<td>0.222</td>
<td>0.322</td>
<td>0.422</td>
<td>0.422</td>
<td>0.322</td>
<td>0.222</td>
<td>0.122</td>
<td>0.122</td>
</tr>
<tr>
<td>$L_{10}$</td>
<td>0.022</td>
<td>0.022</td>
<td>0.122</td>
<td>0.122</td>
<td>0.222</td>
<td>0.322</td>
<td>0.322</td>
<td>0.222</td>
<td>0.122</td>
<td>-3.020</td>
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</tr>
<tr>
<td>$L_{11}$</td>
<td>0.022</td>
<td>0.022</td>
<td>0.122</td>
<td>0.122</td>
<td>0.222</td>
<td>0.322</td>
<td>0.322</td>
<td>0.222</td>
<td>0.122</td>
<td>0.022</td>
<td>-17.113</td>
</tr>
</tbody>
</table>

\[ \sum_{i \neq j} E_{w_i, L_j} = 0.000 \]

Table 8D. Partial factor price (wage) elasticities, $E \left( w_i, L_i \right)$, $E \left( w_i, L_j \right)$ by (63) and Table 4.

<table>
<thead>
<tr>
<th></th>
<th>$L_1$</th>
<th>$L_2$</th>
<th>$L_3$</th>
<th>$L_4$</th>
<th>$L_5$</th>
<th>$L_6$</th>
<th>$L_7$</th>
<th>$L_8$</th>
<th>$L_9$</th>
<th>$L_{10}$</th>
<th>$L_{11}$</th>
<th>$\sum_{i \neq j} E_{w_i, L_j}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$w_1$</td>
<td>-0.200</td>
<td>0.001</td>
<td>0.008</td>
<td>0.013</td>
<td>0.031</td>
<td>0.050</td>
<td>0.050</td>
<td>0.030</td>
<td>0.015</td>
<td>0.001</td>
<td>0.001</td>
<td>0.000</td>
</tr>
<tr>
<td>$w_2$</td>
<td>0.000</td>
<td>-0.199</td>
<td>0.008</td>
<td>0.013</td>
<td>0.031</td>
<td>0.050</td>
<td>0.050</td>
<td>0.030</td>
<td>0.015</td>
<td>0.001</td>
<td>0.001</td>
<td>0.000</td>
</tr>
<tr>
<td>$w_3$</td>
<td>0.001</td>
<td>0.004</td>
<td>-0.286</td>
<td>0.024</td>
<td>0.045</td>
<td>0.066</td>
<td>0.065</td>
<td>0.044</td>
<td>0.027</td>
<td>0.008</td>
<td>0.001</td>
<td>0.000</td>
</tr>
<tr>
<td>$w_4$</td>
<td>0.001</td>
<td>0.004</td>
<td>0.014</td>
<td>-0.276</td>
<td>0.045</td>
<td>0.066</td>
<td>0.065</td>
<td>0.044</td>
<td>0.027</td>
<td>0.008</td>
<td>0.001</td>
<td>0.000</td>
</tr>
<tr>
<td>$w_5$</td>
<td>0.001</td>
<td>0.007</td>
<td>0.021</td>
<td>0.035</td>
<td>-0.341</td>
<td>0.082</td>
<td>0.081</td>
<td>0.058</td>
<td>0.039</td>
<td>0.015</td>
<td>0.003</td>
<td>0.000</td>
</tr>
<tr>
<td>$w_6$</td>
<td>0.002</td>
<td>0.011</td>
<td>0.027</td>
<td>0.046</td>
<td>0.074</td>
<td>-0.403</td>
<td>0.096</td>
<td>0.072</td>
<td>0.051</td>
<td>0.021</td>
<td>0.004</td>
<td>0.000</td>
</tr>
<tr>
<td>$w_7$</td>
<td>0.002</td>
<td>0.011</td>
<td>0.027</td>
<td>0.046</td>
<td>0.074</td>
<td>0.097</td>
<td>-0.404</td>
<td>0.072</td>
<td>0.051</td>
<td>0.021</td>
<td>0.004</td>
<td>0.000</td>
</tr>
<tr>
<td>$w_8$</td>
<td>0.001</td>
<td>0.007</td>
<td>0.021</td>
<td>0.035</td>
<td>0.059</td>
<td>0.082</td>
<td>0.081</td>
<td>-0.342</td>
<td>0.039</td>
<td>0.015</td>
<td>0.003</td>
<td>0.000</td>
</tr>
<tr>
<td>$w_9$</td>
<td>0.001</td>
<td>0.004</td>
<td>0.014</td>
<td>0.024</td>
<td>0.045</td>
<td>0.066</td>
<td>0.065</td>
<td>0.044</td>
<td>-0.273</td>
<td>0.008</td>
<td>0.001</td>
<td>0.000</td>
</tr>
<tr>
<td>$w_{10}$</td>
<td>0.000</td>
<td>0.001</td>
<td>0.008</td>
<td>0.013</td>
<td>0.031</td>
<td>0.050</td>
<td>0.050</td>
<td>0.030</td>
<td>0.015</td>
<td>-0.199</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>$w_{11}$</td>
<td>0.000</td>
<td>0.001</td>
<td>0.008</td>
<td>0.013</td>
<td>0.031</td>
<td>0.050</td>
<td>0.050</td>
<td>0.030</td>
<td>0.015</td>
<td>-0.200</td>
<td>0.000</td>
<td>0.000</td>
</tr>
</tbody>
</table>

\[ \sum_{i \neq j} E_{w_i, L_j} = 0.000 \]
9 Appendix B: CRESH complementarity elasticities

A1. The partial complementarity elasticity, \( c_{ij} \), between any factor pair \((L_i, L_j)\) within in the implicit CRESH function, \( L_A = f(L_1, L_2, ..., L_M) \), (10), was defined in (56) as:

\[ c_{ij} \equiv c_{ij}(L_1, L_2, ..., L_M) = \frac{\partial^2 f}{\partial L_i \partial L_j} ; \quad i = 1, ..., M, \ j = 1, ..., M; \quad c_{ii} \equiv c_{ii}(L_1, L_2, ..., L_M) = \frac{\partial^2 f}{\partial L_i^2} \] (67)

The first-order derivatives in (67) were already given in (12) as,

\[ \forall L_i > 0 : \frac{\partial f}{\partial L_i} - \frac{\partial F}{\partial L_i} = \frac{\partial F}{\partial L_A} = \frac{\alpha_i \rho_i \left( L_i / L_A \right)^{\rho_i - 1}}{\sum_{i=1}^{M} \alpha_i \rho_i \left( L_i / L_A \right)^{\rho_i}} > 0, \quad i = 1, ..., M \] (68)

The second-order derivatives in (67) are derived from the second term (ratio) in (68) as,

\[ \frac{\partial^2 f}{\partial L_i \partial L_j} = \frac{-1}{\left( \frac{\partial f}{\partial L_A} \right)^3} \left[ \frac{\partial^2 F}{\partial L_i \partial L_j} \left[ \frac{\partial F}{\partial L_A} \right]^2 - \frac{\partial^2 F}{\partial L_i \partial L_A} \frac{\partial F}{\partial L_A} - \frac{\partial^2 F}{\partial L_j \partial L_A} \frac{\partial F}{\partial L_A} + \frac{\partial^2 F}{\partial L_i \partial L_j} \frac{\partial F}{\partial L_A} + \frac{\partial^2 F}{\partial L_i \partial L_A} \frac{\partial F}{\partial L_A} + \frac{\partial^2 F}{\partial L_j \partial L_A} \frac{\partial F}{\partial L_A} \right] \] (69)

\[ \frac{\partial^2 f}{\partial L_i^2} = \frac{-1}{\left( \frac{\partial f}{\partial L_A} \right)^2} \left[ \frac{\partial^2 F}{\partial L_i^2} \left[ \frac{\partial F}{\partial L_A} \right]^2 - 2 \frac{\partial^2 F}{\partial L_i \partial L_A} \frac{\partial F}{\partial L_A} + \frac{\partial^2 F}{\partial L_i \partial L_A} \frac{\partial F}{\partial L_A} \right] \] (70)

Insert first-order and second-order derivatives (68-70) of \( L_A = f(L_1, L_2, ..., L_M) \) into (67):

\[ c_{ij} = -L_A \left[ \frac{\partial^2 F}{\partial L_i \partial L_j \partial L_A} - \frac{\partial^2 F}{\partial L_i \partial L_A} - \frac{\partial^2 F}{\partial L_j \partial L_A} + \frac{\partial^2 F}{\partial L_i \partial L_A} \frac{\partial F}{\partial L_A} + \frac{\partial^2 F}{\partial L_j \partial L_A} \frac{\partial F}{\partial L_A} \right] \] (71)

In CRESH cases, we have, \( \frac{\partial^2 F}{\partial L_i \partial L_j} = 0 \), if \( i \neq j \), cf. (74-75). Hence (71) becomes:

\[ c_{ij} = -L_A \left[ -\frac{\partial^2 F}{\partial L_i \partial L_A} - \frac{\partial^2 F}{\partial L_j \partial L_A} + \frac{\partial^2 F}{\partial L_i \partial L_A} \frac{\partial F}{\partial L_A} + \frac{\partial^2 F}{\partial L_j \partial L_A} \frac{\partial F}{\partial L_A} \right], \quad i \neq j \] (72)

\[ c_{ii} = -L_A \left[ -\frac{\partial^2 F}{\partial L_i^2} - 2 \frac{\partial^2 F}{\partial L_i \partial L_A} + \frac{\partial^2 F}{\partial L_i^2} \frac{\partial F}{\partial L_A} \right], \quad i = j \] (73)

To obtain explicit CRESH formulas from (72-73), the parametric expressions of the first-order and second-order derivatives of the CRESH function, \( F(L_A, L_1, L_2, ..., L_M) \), (7) are now needed. We already have the first-order derivatives of \( F \) as, cf. (11),

\[ \frac{\partial F}{\partial L_i} = \gamma \alpha_i \rho_i \left( L_i / L_A \right)^{\rho_i - 1} / L_A \equiv \frac{\gamma \varepsilon_i \beta}{L_i}, \quad \frac{\partial F}{\partial L_A} = -\gamma \sum_{i=1}^{M} \alpha_i \rho_i \left( L_i / L_A \right)^{\rho_i} / L_A \equiv -\gamma \beta / L_A \] (74)

where, \( \beta \equiv \sum_{i=1}^{M} \alpha_i \rho_i \left( L_i / L_A \right)^{\rho_i} \); \( \varepsilon_i = \alpha_i \rho_i \left( L_i / L_A \right)^{\rho_i} / \beta, \ i=1, ..., M \), cf. (14).

The second-order derivatives of \( F \) are derived from the second terms (ratios) in (74) as,

\[ \frac{\partial^2 F}{\partial L_i \partial L_j} = 0; \quad \frac{\partial^2 F}{\partial L_i \partial L_A} = -\gamma \alpha_i \rho_i^2 L_i^{\rho_i - 1} L_A^{-\rho_i} = -\gamma \rho_i \varepsilon_i \beta / L_i L_A; \quad \frac{\partial^2 F}{\partial L_j \partial L_A} = -\gamma \rho_j \varepsilon_j \beta / L_j L_A \] (75)
\[
\frac{\partial^2 F}{\partial L_i^2} = \frac{1}{L_A} \gamma_i \rho_i (\rho_i - 1) (L_i / L_A)^{\rho_i - 2} \frac{1}{L_i^2} = \frac{\gamma (\rho_i - 1)}{L_i^2} \varepsilon_i \beta \quad (76)
\]
\[
\frac{\partial^2 F}{\partial L_A^2} = \gamma \sum_{i=1}^{M} \alpha_i \rho_i (1 + \rho_i) L_i^{\rho_i} L_A^{-2 - \rho_i} = \gamma \frac{\beta}{L_A} \left[ \sum_{i=1}^{M} (1 + \rho_i) \varepsilon_i \right] = \gamma \frac{\beta}{L_A^2} (1 + \tilde{\rho}) \quad (77)
\]
where, \( \tilde{\rho} = \sum_{i=1}^{M} \varepsilon_i \rho_i \). Finally, inserting (74-77) into (72-73) give,
\[
c_{ij} = -L_A \left[ -\frac{-\gamma \rho_i \varepsilon_i \beta}{L_i L_A} - \frac{-\gamma \rho_i \varepsilon_i \beta}{L_i} + \frac{\frac{\gamma \beta}{L_A} (1 + \tilde{\rho})}{L_i} \right] = -\rho_i - \rho_j + 1 + \tilde{\rho}; \quad \tilde{\rho} = \sum_{i=1}^{M} \varepsilon_i \rho_i \quad (78)
\]
\[
c_{ii} = -L_A \left[ \frac{\gamma (\rho_i - 1) \varepsilon_i \beta}{L_i^2} \left[ \frac{-\gamma \beta}{L_A} \frac{1}{L_i} \right] - \frac{-2 \gamma \rho_i \varepsilon_i \beta}{L_i L_A} + \frac{\frac{\gamma \beta}{L_A} (1 + \tilde{\rho})}{L_i} \right] = \rho_i - 1 \varepsilon_i - 2 \rho_i + 1 + \tilde{\rho} \quad (79)
\]
Labor complementarity elasticities \( c_{ij} \) (78-79) satisfy regularity (summation) property:
\[
\sum_{j=1}^{M} \varepsilon_j c_{ji} = \sum_{j=1, j \neq i}^{M} \varepsilon_j c_{ji} + \varepsilon_i c_{ii} = \sum_{j=1, j \neq i}^{M} \varepsilon_j (1 - \rho_i - \rho_j + \tilde{\rho}) + \varepsilon_i (1 - 2 \rho_i + \rho_i - 1 \varepsilon_i + \tilde{\rho})
\]
\[
= -\sum_{j=1, j \neq i}^{M} \varepsilon_j \rho_j - \rho_i \sum_{j=1, j \neq i}^{M} \varepsilon_j - 2 \rho_i \varepsilon_i + \rho_i + \tilde{\rho} = -\sum_{j=1}^{M} \varepsilon_j \rho_j + \tilde{\rho} = -\tilde{\rho} + \tilde{\rho} = 0 \quad (80)
\]
CRESH complementarity elasticities (78-80) were seen in Table 8C for Denmark (2010).

A2. Wage Income function, \( W(L_1, L_2, \ldots, L_M, W_A) \) - Wage Sum, \( W_A L_A \), defined as,
\[
W(L_1, L_2, \ldots, L_M, W_A) = W_A f(L_1, L_2, \ldots, L_M) = c(w_1, w_2, \ldots, w_M) L_A = \sum_{i=1}^{M} w_i L_i \equiv W_A L_A \quad (81)
\]
From Wage Income function (81), we get the basic dual expressions, cf. (14), (52), (67),
\[
E(W, L_i) = \frac{\partial W}{\partial L_i} L_i = \frac{w_i L_i}{W} = \frac{\partial f}{\partial L_i} L_i = E(W, L_i) = \varepsilon_i = E(L_A, L_i); \quad E(W, W_A) = 1 \quad (82)
\]
\[
\frac{\partial^2 W}{\partial L_i \partial L_j} W = \frac{w_i f W_A}{\partial L_i} = \frac{\partial^2 f}{\partial L_i \partial L_j} = c_{ij}; \quad i = 1, \ldots, M, \quad j = 1, \ldots, M \quad (83)
\]
\[
w_i = w_i (L_1, L_2, \ldots, L_M, W_A) = \frac{\partial W(L_1, L_2, \ldots, L_M, W_A)}{\partial L_i} = W_A \cdot \frac{\partial f(L_1, L_2, \ldots, L_M)}{\partial L_i} \quad (84)
\]
where \( w_i \) (84) is the "shadow value" (marginal value-added : \( W_A \cdot \frac{\partial f}{\partial L_i} \)) of one unit increase of specific Labor inputs from age-group \( i \), \( L_i \), i.e., \( w_i \) is the nominal factor price (money annual wage) of \( L_i \) - being obtained as Inverse factor demand price or named 'partial market equilibrium' wage for the Labor supply of Age \( i \), \( L_i \) - with fixed Labor supplies of all the other Age groups - and with a fixed aggregate (average) annual wage, \( W_A \), to
the Total (Aggregate) Labor market equilibrium [complying with full general equilibrium of competitive product and factor markets].

Finally, using (82-83), we shall derive the annual partial wage elasticities of the optimal (Pareto efficient) annual wages \((w_i)\), \([Inverse factor demands], (84)\), with respect to partial variation of own labor supply \((L_i)\) and any cross labor supply \((L_j)\), i.e.,

\[
\frac{\partial w_i (L_1, L_2, ..., L_M, W_A)}{\partial L_j} = \frac{\partial^2 W (L_1, L_2, ..., L_M, W_A)}{\partial L_i \partial L_j} \quad ; \quad i = 1, ..., M , \quad j = 1, ..., M \quad (85)
\]

\[
E(w_i, L_j) = \frac{\partial w_i L_j}{\partial L_j w_i} = \frac{\partial^2 W}{\partial L_i \partial L_j} \quad \frac{L_j}{w_i} = \frac{\partial^2 W}{\partial L_i \partial L_j} \quad \frac{L_j}{w_i} = \frac{w_i L_j}{W} = \varepsilon_j c_{ij} \quad (86)
\]

\[
E(w_j, L_i) = \frac{\partial w_j L_i}{\partial L_i w_j} = \frac{\partial^2 W}{\partial L_j \partial L_i} \quad \frac{L_i}{w_j} = \frac{\partial^2 W}{\partial L_j \partial L_i} \quad \frac{L_i}{w_j} = \frac{w_j L_i}{W} = \varepsilon_i c_{ij} \quad (87)
\]

\[
E(w_i, L_i) = \frac{\partial w_i L_i}{\partial L_i w_i} = \frac{\partial^2 W}{\partial L_i^2} \quad \frac{L_i}{w_i} = \frac{\partial^2 W}{\partial L_i^2} \quad \frac{L_i}{w_i} = \frac{w_i L_i}{W} = \left(\frac{\partial^2 W}{\partial L_i^2}\right)^2 = \varepsilon_i c_{ii} \quad (88)
\]

where complementarity elasticities, \(c_{ij}\), (78-79), are the relevant numbers for obtaining the basic partial wage elasticities, (63), (86-88), that are involved in our demographic population (cohort) impact analyses (calculations) over the projection period, 2020-2090.

Annual wage elasticities of Labor supply (63),(86-88) have summation properties, as (65):

\[
(i) \quad \sum_{j=1}^{M} E (w_i, L_j) = \sum_{j=1}^{M} \varepsilon_j c_{ji} = 0 \quad ; \quad (ii) \quad \sum_{j=1}^{M} \varepsilon_j E (w_j, L_i) = 0 \quad (89)
\]

Annual wage elasticities (63), (86-89), were illustrated in Table 8D for Denmark (2010).

By the way, the ”adding-up”, summing-property (89,i) is easily understood to hold from the ”shadow-value” (wage) functions (84) being homogeneous of degree zero in increasing all labor supplies - by derivatives of the Wage Income function and Aggregator function, (81), (being homogeneous of degree one in labor supplies). Increasing proportionally all labour supplies does not change the relative wages - hence economically (89, i).

Actual checking (89, ii) for CRESH (80) was more cumbersome ; but this was of course necessary for the CRESH formula demographic-labor applications in sections 4-5.

9.1 Price functions, Inverse demands - Hotelling-Wold identity

9.1.1. Existence of consumer good price functions. With regular (monotone, quasiconcave, smooth) Utility functions, \(u = U (q_1, .., q_n)\), and Budget constraint, \(P_1q_1 + .. + P_nq_n = C\),
there is one and only one set of consumer good prices, \( P_i, i = 1, 2, ..., n \), for which exogenously fixed quantities, \((q_1, ..., q_n)\), are optimal (max.\( U \)); this price set is given by:

\[
P_i \frac{C}{C} = \varphi_i(q_1, ..., q_n) = \frac{\partial U}{\partial q_1} q_1 + \frac{\partial U}{\partial q_2} q_2 + ... + \frac{\partial U}{\partial q_n} q_n ; \quad P_i = \varphi_i C, \quad i = 1, 2, ..., n \tag{90}
\]


Given explicit, \( u = U(q_1, ..., q_n) \), the functions, \( \varphi_i(q_1, ..., q_n) \), (90), are easily obtained.

9.1.2. Existence of factor price functions. Given a regular (monotone, concave, smooth, homogeneous of degree one) Production function, \( Y = g(x_1, ..., x_m) \), generating Total revenue (Factor income, Value-added), \( V \equiv PY = w_1x_1 + ..., + w_mx_m \), there is one and only one set of factor prices, \( w_i, i = 1, 2, ..., n \), for which the factor quantities, \((x_1, ..., x_n)\), are optimal (maximizing profit); this factor price set is given by:

\[
\frac{w_i}{P_Y} = \psi_i(x_1, ..., x_m) = \frac{\partial g}{\partial x_1} x_1 + \frac{\partial g}{\partial x_2} x_2 + ... + \frac{\partial g}{\partial x_m} x_m ; \quad w_i = P \frac{\partial g}{\partial x_1} (x_1, ..., x_m), \quad i = 1, 2, ..., m \tag{91}
\]

\(w_i\) - factor price functions, RHS, (91), are the money value marginal factor productivity equations - or the Inverse factor demand functions [for competitive general equilibrium in both product (fixed \( P \)) and factor markets (fixed supply of other factors, \( x_j, j \neq i \)].

9.1.3. Existence of Age annual wage functions. Given a regular (monotone, concave, smooth, homogeneous of degree one) Labor Aggregator function, \( L_A = f(L_1, L_2, ..., L_M) \), giving Total wage income (Labor earnings), \( W \equiv W_AL_A = w_1L_1 + ..., + w_ML_M \), there is only one set of annual wages, \( w_i, i = 1, 2, ..., n \), for which Labor supplies of Age groups, \((L_1, L_2, ..., L_M)\), are used efficiently (maximizing Total wages); this wage set is given by:

\[
\frac{w_i}{W_{AL_A}} = \psi_i(L_1, ..., L_M) = \frac{\partial f}{\partial L_1} L_1 + \frac{\partial f}{\partial L_2} L_2 + ... + \frac{\partial f}{\partial L_M} L_M ; \quad w_i = W_{AL_A} \frac{\partial f}{\partial L_i} (L_1, ..., L_M), \quad i = 1, 2, ..., M \tag{92}
\]

\(w_i\) - annual wage functions, RHS, (92), are the money value marginal Labor contributions of \( L_i \) to Wage sum, \( W \) - or Inverse Labor demand functions [for competitive equilibrium of the Aggregate Labor market (fixed \( W_A \)) and fixed Labor supply of other ages, \( L_j, j \neq i \)].

RHS, (93), shows that wage functions \( w_i \) (92) meet Total Wage (\( W \)) accounting identity.

CRESH implementations of (92-93) are seen as Age-wage profiles, \( w_i, i=1,..,M \), in (25-27), (84), \( w_{IL} \), (30), (33), and to partial wage elasticities in (85-89). Using RHS (91) gives Macro wages, \( W_{AJ}, J=I,II \), (37-38), by CRESH production function (34-36).\(^{10}\)

\(^{10}\)Hotelling calls: \( P_i = \varphi_i(q_1, ..., q_n) C, \quad i = 1, ..., n \), demand functions; see also Hotelling (1932, p.590).
10 App.C: Canonical Wage Structure Model - CRESH

Card and Lemieux (2001, pp.709) used two CES Subaggregators, $L_{AI}$: College Labor, $C = I$, and High-school Labor, $H = II$ - stated in notation, (9), (30), and $\rho_J = \rho$,

$$L_{AI} : \frac{L_{AI}}{L_{AI}} = \left[ \sum_{i=1}^{M} \alpha_{iJ} L_{iJ}^\rho \right]^\frac{1}{\rho} , \quad L_{AI} = \left[ \sum_{i=1}^{M} \alpha_{iJ} L_{iJ}^\rho \right]^\frac{1}{\rho} : -\infty < \rho \leq 1, \quad \sigma = \frac{1}{1 - \rho} \quad (94)$$

As in existing literature, Aggregate output (Y) comes with CES function of $L_{AI}$, $L_{AI}$:

$$Y = \left[ \sum_{J=I}^{II} a_j(t) L_{AI}^\rho \right]^\frac{1}{\sigma} = \frac{\partial Y}{\partial L_{AI}} = \frac{\partial Y}{\partial L_{AI}} : \frac{\partial Y}{\partial L_{AI}} = \frac{\partial Y}{\partial L_{AI}} , \quad \frac{w_{iJ}}{w_{IIJ}} = \frac{\partial Y}{\sigma \partial L_{AI}} = \frac{\partial Y}{\partial L_{AI}} \quad (95)$$

The marginal product (output) of workers in age group (i) - with College (I) or High School education (II) - are seen (by chain rule) in (95). Pareto efficient utilization of different labor qualities (I, II) requires that relative wages, $\frac{w_{I}}{w_{IIJ}}$, are equated to relative marginal products, RHS, (95). The partial derivatives of the CES functions in (94) and (95) imply that relative wages in same age group (i), $\frac{w_{I}}{w_{IIJ}}$, satisfy equation (96),

$$\frac{w_{I}}{w_{IIJ}} (t) = \frac{a_I(t) \alpha_{I}}{a_I(t) \alpha_{I}} \left[ L_{AI} (t) \right]^\rho - \left[ L_{I} (t) \right]^\rho - 1 ; \rho_{s} = -1, \quad \rho_{s} + 1 = \frac{1}{\sigma} , \quad \rho_{s} - 1 = -1 \quad (96)$$

which is equivalent to expression (97) - with Employment (Supply)32 ratios (Labor proportion of College/High school workers), and using Age composition (distribution) within College/High school workers ($\lambda_{I,J}$), (97). Evidently from (97), larger substitution elasticities ($\sigma_{s}$ and $\sigma$) imply smaller changes in the relative wages, $\frac{w_{I}}{w_{IIJ}}$ [or log changes $r_i(t)$, (98)], coming from variation in Aggregate Supply ratios and Age compositions ($\lambda_{I,J}$).

$$r_i(t) \equiv \log \frac{w_{I}}{w_{IIJ}} (t) , \quad i = 1 = 26 - 30, 31 - 35, ..., M = 7 = 56 - 60 ; \quad I, II \quad (98)$$

31Card and Lemieux (2001, p.710, equation 7) presents (96), LHS, in logarithmic form, which is more convenient for parameter estimation purposes. We do not enter estimation - will only discuss the results.

32In relative wages (97), the aggregate supply ratio (relative supplies) is also seen in share form, $\lambda_{AI}$:

$$\frac{w_{I}}{w_{IIJ}} (t) = \frac{a_I(t) \alpha_{I}}{a_I(t) \alpha_{I}} \left[ \frac{\lambda_{AI}}{\lambda_{AIJ}} (t) \right]^{-\frac{1}{\rho}} \left[ \frac{\lambda_{I}}{\lambda_{IIJ}} (t) \right]^{-\frac{1}{\rho}}, \quad \lambda_{AI} = \frac{L_{AI}}{L_{AIJ}}, \lambda_{AIJ} = \frac{L_{AIJ}}{L_{AIJ}}, \sum_{J=I}^{II} \lambda_{AIJ} = 1$$

For changes in log shares - $M_{AIJ}^J$ - 100 - of aggregate labor input groups and their relative wages changes, see Katz and Murphy (1992, p.39-40,49,67-68) - where the aggregate labor supplies, $L_{AI}$, are measured in so-called ‘efficiency units’. No Age-specific full-time equivalents $L_{AI}$ appear in Katz & Murphy (1992).
For year \( (t) \), \( r_i(t) \) is called a College-High school premium or wage gap for age group \((i)\). Ratios \( r_i(t) \), \( i = 1, .., 7 \) is an Age profile \((98)\) of premiums/wage gaps for calendar year \((t)\).


For all tree countries, CES parameter estimates, Card & Lemieux (2001, pp.725-27), of \( \rho_s \), \((94)\), \((96)\), were in the range : \( \rho = 0.77 \) to \( \rho = 0.83 \) \(^{33}\), i.e., \( \sigma = 4.34 \) to \( \sigma = 5.88 \).

The estimated sizes of the age specific efficiency (intensity) parameters - \( \alpha_{si} \) and \( \alpha_{sii} \) - in \((94)\), \((96-97)\), for the seven \((98)\) age groups, \( i = 1, .., M=7 \), are not available \(^{34}\) (reported); they give the two age-wage profiles of, \( w_{i1}(t) \), \( w_{i11}(t) \), \( i = 1, ,M \), by \((94)\) - as seen below.

In contrast to \( \alpha_{s1} \), \( \alpha_{s11} \), the relative efficiency (intensity) parameters, \( a_t, a_{11} \), \((95-97)\), are not time-invariant for the aggregates \( L_{A1} \) and \( L_{A11} \) in the Production function \((95)\).

Card & Lemieux (p.725) give for the tree countries estimates of year effects, reflecting changes (technology shocks) to the ratios, \( \frac{a_t(t)}{a_{11}(t)} \), in the calendar years above. These year effects are rising, and next replaced by linear trends : \( \frac{a_t(t)}{a_{11}(t)} = \beta t, \beta \in (0.017,0.020) \) for US, \( \beta \in (0.021,0.018) \) for UK, \( \beta \approx 0 \) for Canada. These trend estimates \((\beta)\) are combined with the final estimation of \( \rho_s \), \((95)\). Card & Lemieux (p.727) present two estimates of \( \rho_s \) [depending on sizes of the Aggregate Supply indexes for College Labor and High-school Labor, 1. Katz-Murphy indexes, 2. \( L_{A1} \) in \((94)\)]. Thus we see the estimates for the US:

1. \( \rho_s = 0.59 \), 2. \( \rho_s = 0.52 \), i.e., \( \sigma_s = 2.44 \) or \( \sigma_s = 2.08 \). For UK: 1. \( \rho_s = 0.53 \), 2. \( \rho_s = 0.66 \), i.e., \( \sigma_s = 2.13 \) or \( \sigma_s = 2.94 \). For Canada: 1. \( \rho_s = 0.93 \), 2. \( \rho_s = 0.87 \), i.e., \( \sigma_s = 14.29 \) or \( \sigma_s = 7.69 \) (Canadian \( \rho_s \) are imprecise estimates, also \( \rho_s = 0.82, \sigma_s = 3.57 \).

By using second estimate \((2)\) of \((\rho_s, \sigma_s)\), second trend coefficient of \((\beta)\), common estimate of \((\rho_t, \rho_{11} = \rho, \sigma)\), together with relative Aggregate Labor Supplies, \( L_{A1}(t)/L_{A11}(t) \), and relative Age-group Supplies, \( L_{i1}(t)/L_{i11}(t) \), [relative Age distributions, \( \lambda_{i1}(t)/\lambda_{i11}(t) \)] we obtain with \((96-97)\) their Age-relative wage profile, \( w_{i1}(t)/w_{i11}(t) \), \( i = 1, ,M = 7 \), for

\(^{33}\) In Table 4 (Col.3), the range of the CRESH age-specific \( \rho_i \) was : \( \rho_i = 0.5 \) \( \rho_i = 0.8 \); but \( \rho_i = 0.5 \) applied only to two age groups, 40-44, 45-49. CES parameter \( \rho = 0.8 (\sigma = 5) \) can fit any data of relative wages pretty well; cf. Guest & Jensen (2016, p.31, Fig.4 - misprint p.32: interchange titles of Fig 5,6).

\(^{34}\) Card & Lemieux (p.713, eq.12a-12b) show how, \( \alpha_{s1}, \alpha_{s11}, (s_i, s_j), (\eta), \rho (\eta) \), are estimated - and used to construct estimates of Aggregate Labor supplies, \( L_{A1}(t), L_{A11}(t) \), \( (C_t, H_t) \), in \((94)\); cf. footnote 26.
the calendar year (t) - or in log version, Age-relative wage profile (98), \( r_i(t) \), \( i = 1, \ldots, M = 7 \), called the (College premiums - wage gaps) for calendar year (t).

The shape of Age-relative wage profiles, \( \frac{w_{i1}(t)}{w_{iH}(t)} \), [or \( r_i(t) \)], \( i = 1, \ldots, M \), have changed over years 1970,1975,1980,1985,1990,1995 - but alike for three countries, Card & Lemieux (p.718). It has shifted upwards before 1980 - and rotated (twisted) after 1980-85, with younger workers (31-35, 36-40) rising much more than the older workers (46-50, 51-55).

Apart from 'biased' technology trends, \( a_i(t)/a_{iH}(t) = \beta t \), Card & Lemieux (p.707) see a deceleration (slower increases) in relative College Labor supplies since 1980 as the driving force behind the increased \(^{35}\) relative wages (96-98). But behind such relative wages, \(^{36}\) (96-98), we also want economic-analytically to know exactly what occur - in a consistent way - to the corresponding calendar year (t) Age-wage profiles : \( w_{i1}(t) \), \( i = 1, 2, \ldots, M \), and, \( w_{iH}(t) \), \( i = 1, 2, \ldots, M \). We will match CES version of \( w_{i1}(t) \), (30), to (94-97).


\[^{36}\]As background for CRESH production function, (34-36), the general CES version is discussed here. The CES production functions are (normalizing, \( \sum_{j=1}^{VI} a_j = 1 \), may need \( \gamma \) for dimensional reasons):

\[
Y = \gamma \left( \sum_{j=1}^{VI} a_j (b_j X_j) \right) \gamma ; \quad \gamma > 0 ; \quad \sum_{j=1}^{VI} a_j = 1 ; \quad -\infty < \rho \leq 1 , \quad \sigma = \frac{1}{1-\rho}
\]

where \( b_j \) are factor-augmenting technology terms - parameters (constants) or specified functions of time (trends). The expressions, \( b_j X_j \), are referred to as the factor supplies shown (measured) in "efficiency units". Usually, the variables (quantities), \( X_j, J = 1, \ldots, VI \), have their own units of measurements (e.g., labor, capital, etc.), which are entirely different matters (and problems) than attaching factor-augmenting terms (parameters, functions, \( a_j \)) to each variable \( X_j \) in discrete or continuous time. For our purposes, we did neither consider any factor augmenting terms \( (b_j) \) involved in \( L_{\lambda_j} \), (95), nor \( (b_{ij}) \) in \( L_{i1} \), (94).

Autor et al. (1998, p.1176-79) use such CES two-factor labor augmenting \( (b_j, b_{ij}) \) version of (95), giving the relative wages (ratios of marginal products of two labor types) as follows [notation, (96-97)]:

\[
\frac{w_i(t)}{w_{i1}(t)} = \frac{a_i(t)}{1-a_i(t)} \left( \frac{L_{i1}(t)}{L_{i1}(t)} \right)^\rho \left( \frac{L_i(t)}{L_{i1}(t)} \right)^{-\frac{1}{\rho}} \equiv d(t) \left( \frac{L_i(t)}{L_{i1}(t)} \right)^{-\frac{1}{\rho}} ; \quad \rho = \frac{\sigma - 1}{\sigma} , \quad \sigma = \sigma_y , \quad \rho = \rho_y
\]

\[
w_i = \frac{\partial Y}{\partial X_i} = a_i(t) b_i(t) \left( \frac{Y}{L_i} \right)^{\rho - 1} \quad w_{i1} = \frac{\partial Y}{\partial L_{i1}} = a_{i1}(t) b_{i1}(t) \left( \frac{Y}{L_{i1}} \right)^{\rho - 1} \quad \text{Such technology term} \left( \frac{b_i(t)}{b_{i1}(t)} \right)^\rho \text{may be included in (101). No Age-specific full-time equivalents, } L_{i1}, \text{ appear in Autor et al. (1998)}
\]

The 'parameter elements' within the composite variable, \( d(t) \), are used to reflect technological- and relative factor demand shifts that may favor college equivalents, \( L_i(t) \), raising the college premium/wage gap : \( r(t) \equiv \log \left( \frac{w_{i1}(t)}{w_{iH}(t)} \right) \). Using Katz & Murphy (1992, p.69) point estimate of \( \sigma_y = 1/0.709 = 1.41 \), and lower/upper limits of \( \sigma_y (1, 2) \), Autor et al. (1998), calculate - for \( \sigma_y = 1, 1.4, 2 \) - and with data, \( \frac{L_i(t)}{L_{i1}(t)} \), the college premium, \( r(t) \), and implied relative demand shifts, \( d(t) \), for decades in period 1940-1996.
In (37-38), the *exogenous* $W_{A_j}$ referred to *Average wage* of particular Subaggregates of workers, $L_{A_j}$. With a single-sector (output), aggregate CES production function, (95), the efficiency in production implies that $W_{A_j}$ are *marginal value products of Labor*, i.e.,

$$\frac{W_{A_j}}{P} = \frac{\partial Y}{\partial L_{A_j}} = a_j(t) \left[ \frac{Y}{L_{A_j}} \right]^{\rho_j - 1} ; \quad \frac{W_{A_{11}}}{P} = \frac{\partial Y}{\partial L_{A_{11}}} = a_{11}(t) \left[ \frac{Y}{L_{A_{11}}} \right]^{\rho_{11} - 1} ; \quad Y = \left[ \sum_{j=1}^{11} a_j(t) L_{A_j}^{\rho_j} \right]^{\frac{1}{\rho}}$$

Thus with (99) and CES (94), *calendar wages* of Age group (i), $w_{i,j}(t)$, (30), becomes:

$$w_{i,j}(t) = P \cdot a_{i}(t) \left[ \frac{Y}{L_{A_j}} \right]^{\rho_j - 1} \alpha_{i,j} \lambda_{i,j}^{\rho_j - 1}(t) ; \quad w_{i,11}(t) = P \cdot a_{11}(t) \left[ \frac{Y}{L_{A_{11}}} \right]^{\rho_{11} - 1} \alpha_{11} \lambda_{11}^{\rho_{11} - 1}(t) ; \quad i = 1,...,M$$

Accordingly, (100) is consistent with *relative* wages, LHS (101) - formally similar to (97):

$$\frac{w_{i,j}(t)}{w_{i,11}(t)} = \frac{a_{i}(t) \alpha_{i,j} \lambda_{i,j}^{\rho_j - 1}(t)}{a_{11}(t) \alpha_{11} \lambda_{11}^{\rho_{11} - 1}(t)} \left[ \frac{L_{A_j}(t)}{L_{A_{11}}(t)} \right]^{\frac{1}{\rho_j}} \left[ \frac{\lambda_{i,j}(t)}{\lambda_{11}(t)} \right]^{-\frac{1}{\rho}} , \quad L_{A_j} = L_{j} = \sum_{i=1}^{M} L_{i,j} ; \quad L_{A_{11}} = L_{11} = \sum_{i=1}^{M} L_{i,11}$$

*Note*. In (99-101), *Aggregate Labor supplies* ($L_{A_j}$, $L_{A_{11}}$) are *Age-group sums* (RHS (101)).

We now need to scrutinize the concepts and the actual numbers of the *Aggregate Labor supply* variables, $L_{A_j}$, $j=1,11$, that appear in the CES functions, (94-95). The CES *Subaggregator formulas* (94) are often called *Aggregate Supply indexes* of College/High school Labor resources. Clearly, larger values of CES parameter $\rho$ ($\sigma$) affect the isoquant maps of (94) analogously to the role of $\rho_y$ ($\sigma_y$) for the isoquant maps of (95), implying *e.g.*, that *larger* supply index numbers $L_{A_j}$ are attained with *smaller* sizes of $L_{11}$ in (94); but such larger abstract-theoretical $L_{A_j}$ total supply numbers are neither directly observable nor satisfy simple labor accounting identities stated in RHS (101). With well-defined *measuring units* as ‘full-time worker (College/High school) equivalents’, the direct sum (accounting) of sub-groups in RHS (101) are important to satisfy - as in Table 2 (Col.5). In short, Subaggregator formulas (94) are *not* used to obtain (predict) *Total quantities*, $L_{A_j}$, but (94) are used to *generate* sub-group *wages* to form Total Wage Income, (25-27).
Only derivatives (31-32) of Aggregator functions, \( L_{AJ} = f_J \), are used in imputing wages \( w_{i,t} \), (100), to the age-groups (i) of Total Labor supply,\(^{37}\) \( L_{AJ} = \sum_{i=1}^{M} L_{i,J} \), RHS (101).

The size (level) of the Average wages - \( W_{AJ} \) were not explained economically. The CES Production functions (95) attempted such economic explanation of, \( W_{AJ} \), through the marginal products of Labor, (99), \( L_{AJ}, \ J = I, II \), which implied the calendar Age-wage profiles, \( w_{i,t}(t) \), \( i = 1,., M \), (100), of College/High school Labor, \( J = I, II \). Shifting (twisting, rotating) of the two calendar CES Age-wage profiles is seen by (100) to follow from: 1. divergent evolutions of the (sum) Totals,\(^{38}\) \( L_{AJ}(t), \ L_{AJ}(t) \), 2. divergent evolutions of the Age distributions, \( \lambda_{i,t}, \ i = 1,., M \), within the two categories, \( J = I, II \).

With CRESH, CES calendar year Age-wage profiles (100), are replaced by inserting marginal value products of Labor, (99) - into CRESH Age-wage profile formula (33):

\[
L_{AJ} = w_{i,t}(t) = P \cdot a_i(t) \left[ \frac{Y_{-1}}{L_{i,j}} \right]^{\rho_{i,j}} - \sum_{i=1}^{M} \alpha_{i,1} \rho_{i,1} \lambda_{i,t}^{\rho_{i,1}} - \sum_{i=1}^{M} \alpha_{i,2} \rho_{i,2} \lambda_{i,t}^{\rho_{i,2}}
\]

\[\lambda_{i,t} = 1,..,M \] mattered cf. (31-32). However, the total imputed wage sums are: \( W_{AJ} = \sum_{i=1}^{M} w_{i,j} = \sum_{i=1}^{M} w_{i,t} L_{i,t} \equiv w_{i,t} L_{i,t} \equiv W_{J}, \ J = I, II \), i.e., \( L_{AJ} \) as in RHS (101) - and not as \( L_{AJ} \) determined in (94). Thus \( L_{AJ} \) in (95) is neither (94), but RHS (101) - no ages are involved in the factor substitutions by (95). The text above (p.42) mentioned two estimates of \( (\rho_{i,j}, \sigma_j) \) by: 1. Katz-Murphy indexes, 2. \( L_{AJ} \) in (94). Thus, (\( \rho_{i,j}, \sigma_j \)) estimates by the latter Labor supply numbers (94) are inadequate - instead we wanted estimates of \( \alpha_{i,j} \), cf. footnote 23.

\(^{37}\)Although relative wages in LHS (101) formally look ‘similar’ to (97), there is an ambiguity about the numerical size of, \( L_{AI}, \ L_{AI} \), appearing in Sub-aggregators, (94), Production function (95), and relative wages, (96-97). As discussed above, the absolute size of, \( L_{AI}, L_{AI} \), are irrelevant for using Sub-aggregators, (94) to age-wage imputations, where only age distributions, \( \lambda_{i,t}, \ i = 1,..,M \), mattered cf. (31-32). However, the total imputed wage sums are: \( W_{AJ} = \sum_{i=1}^{M} w_{i,J} = \sum_{i=1}^{M} w_{i,J} L_{i,J} \equiv L_{i,J} \equiv W_{J}, \ J = I, II \), \( i.e., L_{AJ} \) as in RHS (101) - and not as \( L_{AJ} \) determined in (94). Thus \( L_{AJ} \) in (95) is neither (94), but RHS (101) - no ages are involved in the factor substitutions by (95). The text above (p.42) mentioned two estimates of \( (\rho_{i,j}, \sigma_j) \) by: 1. Katz-Murphy indexes, 2. \( L_{AJ} \) in (94). Thus, (\( \rho_{i,j}, \sigma_j \)) estimates by the latter Labor supply numbers (94) are inadequate - instead we wanted estimates of \( \alpha_{i,j} \), cf. footnote 23.

\(^{38}\)The time path of the ratio, \( \frac{L_{AJ}(t)}{L_{AI}(t)} \) is called “intercohort shifts in the relative supply of highly educated workers”, “intercohort trend in educational attainment”, and \( \lambda_{i,t}(t), \lambda_{i,t}(t), \ i = 1,., M \), are called “differences in age distributions of educational attainment”, Card & Lemieux (p.707). Cohorts means calendar year (t) total Labor supplies, \( L_{AJ}(t), \ J = I, II \). Cohorts in the sense of Labor supplies, \( L_{J}^{*}(T) \), [life-cycle ages (i)], and life time supplies, \( L_{J}^{*}(T) \), cf. (44), are not seen in Card & Lemieux (2001).


\(^{39}\)A pure labor economy model by the production and substitution with only two Labor factors (categories), \( L_{AJ}(t), L_{AI}(t), \ (95) \), is of course an abstraction for explaining, \( W_{AJ} \), (99).
References


